MM-SPACES AND GROUP ACTIONS

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ABSTRACT. These are introductory notes on some aspects of concentration of measure in the presence of an acting group and its links to Ramsey theory [1]).

1. INTRODUCTION

It can be argued that the theory we are interested in (call it theory of *mm*-spaces, the phenomenon of concentration of measure on high-dimensional structures, asymptotic geometric analysis, geometry of large dimensions, ...) has been largely shaped up by three publications. These are: the book by Paul Lévy [Lév], Vitali Milman's new proof of the Dvoretzky theorem [M1], and the paper by Gromov and Milman [Gr-M1] which had set up a framework for systematically dealing with concentration of measure. Significantly, in the two latter papers concentration goes hand in hand with group actions on suitable spaces with metric and measure.

It is also known that concentration of measure and combinatorial, Ramsey-type results have a similar nature and are often found together [M3].

A number of attempts have been made to understand the nature of the interplay between concentration, transformation groups, and Ramsey

theory, cf. papers by Gromov [Gr1], Milman [M2,M3], and some others [A-M,Gl,P2,P3,G-P,Gl-W]. However, it is safe to say that there is still a long way to go towards the full understanding of the picture.

Here we aim at providing a readable introduction into this circle of ideas.

2. Some concepts of asymptotic geometric analysis

DEFINITION 1. A *space with metric and measure,* or an *mm-space,* is a triple (X, d, μ), where d is a metric on a set X and μ is a finite Borel measure on the metric space (X, d). It will be convenient to assume throughout that μ is a probability measure, that is, normalized to one.

DEFINITION 2. The *concentration function* α_X of an *mm*-space $X = (X, d, \mu)$ is defined for non-negative real ε as follows:

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2} & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) : A \subseteq X \text{ is Borel}, \ \mu(A) \geq \frac{1}{2}\} & \text{if } \varepsilon > 0. \end{cases}$$

Here A_ε denotes the ε-neighbourhood (ε-fattening, ε-thickening) of A.

EXERCISE 1. Prove that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to \infty$. (For spaces of finite diameter this is of course obvious.)

DEFINITION 3. An infinite family of *mm*-spaces, $(X_n, d_n, \mu_n)_{n=1}^\infty$, is called a *Lévy family* if the concentration functions α_n of X_n converge to zero pointwise on $(0, \infty)$:

$$\forall \varepsilon > 0, \ \alpha_n(\varepsilon) \to 0 \text{ as } n \to \infty.$$

EXERCISE 2. Prove that the above condition is equivalent to the following. Let $A_n \subseteq X_n$ be Borel subsets with the property that

$$\liminf_{n \to \infty} \mu_n(A_n) > 0.$$

Then

$$\forall \varepsilon > 0, \ \lim_{n \to \infty} \mu_n((A_n)_\varepsilon) = 1.$$

The following are some of the most common examples of Lévy families.

EXAMPLE 1. Unit spheres $\mathbf{S}^n$ in the Euclidean spaces $\mathbf{R}^{n+1}$, equipped with the Euclidean (or geodesic) distances and the normalized Haar measures (that is, the unique rotation-invariant probability measures). This result is due to Paul Lévy [Lév], though his proof, based on the isoperimetric inequality, was only made rigorous much later by Gromov [Gr2]. (Nowadays simpler proofs, using the Brunn–Minkowski inequality, are known, cf. [Gr-M2, Sch].)

EXAMPLE 2. The special orthogonal groups $SO(n)$, equipped with the normalized Haar measure and the uniform operator metric,

$$d(T, S) := \|T - S\|,$$

induced from $\mathcal{B}(\mathbf{R}^n) \cong M_n$. This was established by Gromov and Milman [Gr-M1]. The same argument holds for the special unitary groups.

EXAMPLE 3. The family of finite permutation groups (S_n), equipped with the uniform (normalized counting) measure and the Hamming distance:

$$d(\sigma, \tau) = \frac{1}{n} |\{i: \sigma(i) \neq \tau(i)\}|.$$

The result is due to Maurey [Ma], see also [Ta1].

EXAMPLE 4. The Hamming cubes $\{0, 1\}^n$ equipped with the normalized counting measure and the Hamming distance $d(x, y) = \frac{1}{n} |\{i: x_i \neq y_i\}|$ form a Lévy family [Sch,M-S].

REMARK 1. All of the above are *normal* Lévy families, meaning that the concentration functions α_n admit Gaussian upper bounds:

$$\alpha_n(\varepsilon) \leq C_1 \exp(-C_2 n \varepsilon^2)$$

for some $C_1, C_2 > 0$.

It should be noted that this is not always the case for 'naturally occurring' Lévy families. For instance, the groups $SL(2, \mathbf{F}_p)$, where p are prime numbers, equipped with the normalized counting measure and the word metric given by a fixed system of generators in $SL(2, \mathbf{Z})$, form a Lévy family with $\alpha_p(\varepsilon) \leq C_1 \exp(-C_2 \sqrt{p}\,\varepsilon)$, [A-M, M4]. (Recall in this connection that the n-th prime number $p_n \sim n \log n$.)

REMARK 2. In Example 4, replace $\{0, 1\}$ with any probability measure space, $X = (X, \mu)$. Equip every finite power X^n with the product measure $\mu^{\otimes n}$ and the normalized Hamming distance $d(x, y) = \frac{1}{n} |\{i: x_i \neq y_i\}|$. Unless X is purely atomic, the measures $\mu^{\otimes n}$ are not Borel, and thus X^n aren't even mm-spaces in the sense of our definition. At the same time, if in the definition of the concentration function we only restrict ourselves to measurable subsets A such that A_ε are also measurable, it can be shown that $X^n, n \in \mathbf{N}$ form a Lévy family in a very reasonable sense. (See [Ta1,Ta3] for far-reaching variations.) If anything, this shows that the full formalization of the subject has not yet been achieved and nothing is cast in stone.

Notice that the mm-spaces from the above Examples 1–4 are at the same time (phase spaces of) topological transformation groups, with both metrics and measures being invariant under group actions. In Example 1 it is the action of the orthogonal — or the unitary — group on the sphere, while in Examples 2–4 the groups act upon themselves on the left.

3. A TRANSFORMATION GROUP FRAMEWORK

Here is the idea of what kind of interaction between concentration phenomenon and group actions one should expect. The following example is borrowed from a paper by Vitali Milman [M4].

Suppose a group G acts on an mm-space (X, d, μ) by measure-preserving isometries. Assume that the mm-space X strongly concentrates, that is, the function $\alpha_X(\varepsilon)$ drops off sharply already for small values of ε. Let us assume, for instance, that the concentration is so strong that, whenever $\mu(A) \geq \frac{1}{7}$, the measure of the $\frac{1}{10}$-neighbourhood of A is strictly greater than 0.99. (Cf. Exercise 2.)

If now we partition X into seven pieces, and pick at random one hundred elements $g_1, g_2, \ldots, g_{100} \in G$, then at least one of the pieces, say A, has the property that all one hundred translates, of $\frac{1}{10}$-neighbourhoods of A by our elements g_i have a point, x^*, in common. Equivalently, x^* is 'close' (closer than $\frac{1}{10}$) to each of the one hundred translates of A.

The above effect becomes more pronounced the higher the level of concentration is. Partition a concentrated ('high-dimensional') mm-space into a small number of subsets, and at least one of them is hard to move.

In order to set up a formal framework, we assume all topological spaces and topological groups appearing in this article to be metrizable, for the reasons
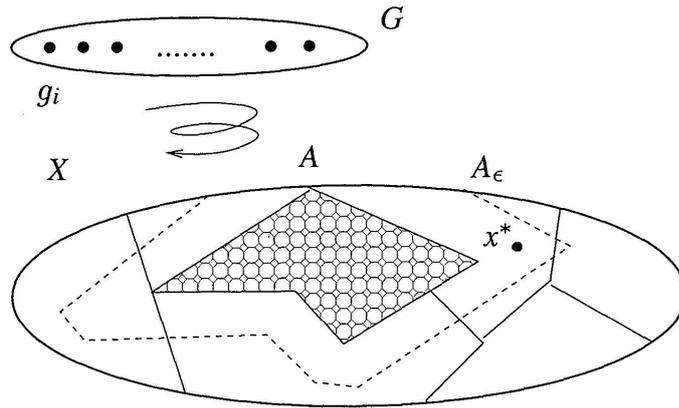
FIGURE 1

Dynamics in the presence of concentration

of mere technical simplicity[2]). We need G-spaces of a particular kind. Let $X = (X, d)$ be a metric space, not necessarily compact, and let a group G act on X (on the left) by *uniformly* continuous maps. In other words, there is a map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, such that $g \cdot (h \cdot x) = (gh) \cdot x$, $e \cdot x = x$, and every map of the form

$$X \ni x \mapsto g \cdot x \in X$$

(a translation by g) is uniformly continuous. (Then it is automatically a uniform isomorphism.) If, moreover, G is a topological group, then we require the action $G \times X \to X$ to be continuous.

EXAMPLE 5. The motivation for our choice of the class of G-spaces is provided by the fact that every (metrizable) compact G-space, K, is such: a translation of K by an element $g \in G$, being a continuous map on a compact space, is uniformly continuous.

Here is another property that compact G-spaces possess automatically, while G-spaces of a more general nature do not.

EXERCISE 3. Let a topological group G act continuously on a (metrizable) compact space $K = (K, d)$. Prove that for every $\varepsilon > 0$ there is a neighbourhood of identity $V \ni e_G$ with the property that whenever $g \in V$ and $x \in K$, one has $d(x, g \cdot x) < \varepsilon$. [In abstract topological dynamics such actions are termed *bounded*, or else *motion equicontinuous*.]

[*Hint.* Using the continuity of the action $G \times K \to K$, choose for each $x \in K$ a neighbourhood U_x of x in K and a neighbourhood, V_x, of e_G in

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[2]) More generally, metrics can be replaced with uniform structures.

G, such that $V_x \cdot U_x \subseteq B_\varepsilon(x)$ (the open d-ball around x); now select a finite subcover of $\{U_x\}, \ldots$]

EXAMPLE 6. Every metrizable group admits a right-invariant compatible metric ($d(x,y) = d(xa, ya)$), as well as a left-invariant one ($d(x,y) = d(ax, ay)$). The action of G on itself by left translations is an action by isometries with respect to a left-invariant metric, and (exercise) an action by uniform isomorphisms with respect to a right-invariant metric.

EXERCISE 4. Show that the action of a topological group G upon itself, equipped with a right invariant metric, by left translations, is bounded.

EXAMPLE 7. One topological group of interest to us is $U(\mathcal{H})_s$, the full unitary group of a separable Hilbert space with the *strong* operator topology. (That is, the topology induced from the Tychonoff product $\mathcal{H}^{\mathcal{H}}$.) A standard neighbourhood of identity in this topology consists of all $T \in U(\mathcal{H})$ such that $\|T(x_i) - x_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$, where $x_1, \ldots, x_n$ is a finite collection of unit vectors in $\mathcal{H}$. This topology on $U(\mathcal{H})$ coincides with the *weak operator topology*, that is, the weakest topology making continuous every map of the form

$$U(\mathcal{H}) \ni T \mapsto \langle x, Tx \rangle \in \mathbf{C}, \quad x \in \mathcal{H}.$$

EXAMPLE 8. Let π be a unitary representation of a group G (viewed as discrete) in a Hilbert space $\mathcal{H}$. Denote by $\mathbf{S}^\infty$ the unit sphere in $\mathcal{H}$, equipped with the norm distance. Then G acts on $\mathbf{S}^\infty$ by isometries: $(g, x) \mapsto \pi_g x$.

REMARK 3. The above G-space is bounded for trivial reasons. It should be noted, however, that in general one does not expect a 'typical' G-space to be bounded at all.

DEFINITION 4. Let a topological group G act continuously, by uniform isomorphisms, on two metric spaces, X and Y. A *morphism*, or an *equivariant map*, from X to Y is a uniformly continuous map $i \colon X \to Y$ which commutes with the action:

$$i(g \cdot x) = g \cdot i(x).$$

DEFINITION 5.   Let a topological group G act continuously on a metric space (X, d) by uniformly continuous maps, and let also G act continuously on a compact space K. Let $i: X \to K$ be a morphism of G-spaces with an everywhere dense image in K. The pair (K, i) is called an *equivariant compactification* of X.

EXAMPLE 9.   Let G and $\mathcal{H}$ be as in Example 8. The unit ball $\mathbf{B}$ in $\mathcal{H}$ equipped with the weak topology is compact, and G acts on $\mathbf{B}$ in the same way as on the sphere. The embedding $\mathbf{S}^\infty \hookrightarrow \mathbf{B}$ is an equivariant compactification.

The following is at the heart of abstract topological dynamics.

THEOREM 1.   *Let G be a topological group, and let d be a right-invariant metric generating the topology of G. Let K be a (metric) compact G-space, and let $\kappa \in K$ be arbitrary. There is a morphism of G-spaces $i: (G, d) \to K$ such that $i(e) = \kappa$.*

Proof.   Define the map $i: G \to K$ (an *orbit map*) by

$$i: G \ni y \mapsto y \cdot \kappa \in K.$$

This map is equivariant. $[i(g \cdot y) = (gy) \cdot \kappa = g \cdot (y \cdot \kappa) = g \cdot i(y).]$ It only remains to check the uniform continuity of i. Choose any continuous metric on K, say ρ. Using Exercise 3, find a $\delta > 0$ with the property that $\rho(x, g \cdot x) < \varepsilon$ whenever $x \in K$ and $d(g, e_G) < \delta$. If now $g, h \in G$ are such that $d(g, h) < \delta$, then $d(gh^{-1}, e_G) < \delta$ and consequently

$$\rho(h\kappa, g\kappa) = \rho(h\kappa, gh^{-1}(h\kappa)) < \varepsilon. \qquad \square$$

REMARK 4.   The difference between the right and left invariant metrics (or, more generally, uniform structures) on a topological group cannot be overemphasized. Even if they are totally symmetric, they cease to be such as soon as we choose the action (in our case, by left translations).

Here is a key notion putting the concentration of measure in a dynamical context.

DEFINITION 6. Let a metrizable topological group G act continuously by uniform isomorphisms on a metric space $X = (X, d)$. Say that the G-space (transformation group) (G, X) is *Lévy* (Gromov and Milman [Gr-M1]) if there are a sequence of subgroups of G

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq G,$$

and a sequence of probability measures

$$\mu_1, \mu_2, \ldots, \mu_n, \ldots$$

on (X, d), such that

(i) $\bigcup G_n$ is everywhere dense in G,

(ii) μ_n are G_n-invariant,

(iii) (X, d, μ_n) form a Lévy family.
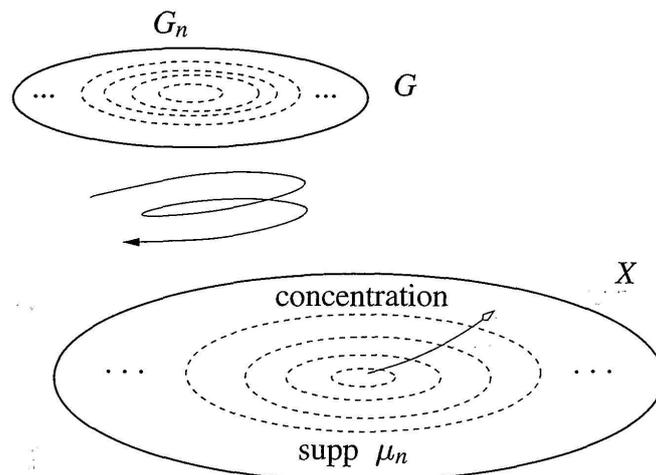
FIGURE 2

A Lévy transformation group

In the particular case where X is the group itself equipped with a right-invariant metric and the action of G is by left translations, we say that G is a *Lévy group*.

EXAMPLE 10. Let $\mathcal{H} = \ell_2$, and let $G = U(\mathcal{H})_s$, $X = (G, d)$, where d is a right-invariant metric and the action is by left translations. Set $G_n = SU(n)$ (embedded into $U(\mathcal{H})$ as a subgroup of block-diagonal operators), and let μ_n denote the normalized Haar measure on $SU(n)$. One can view μ_n as a measure on all of $U(\ell_2)_s$ with support $SU(n)$. The mm-spaces $(U(\mathcal{H})_s, d, \mu_n)$ clearly form a Lévy family, because the spaces $(SU(n)_s, d|_{SU(n)}, \mu_n)$ do. We conclude: $U(\mathcal{H})_s$ is a Lévy group.

EXAMPLE 11. Let π be a strongly continuous unitary representation of a compact group G in ℓ_2. Then ℓ_2 decomposes into the orthogonal direct sum of finite-dimensional (irreducible) unitary G-modules, $\ell_2 \cong \bigoplus_{n=1}^{\infty} V_n$. Set for each $n \in \mathbf{N}$

$$\mathbf{S}_n = \mathbf{S}^{\infty} \cap \bigoplus_{i=1}^{n} V_n .$$

We obtain a nested sequence of spheres of increasing finite dimension which are invariant under the action of G. Let μ_n denote the rotation-invariant probability measure on the sphere $\mathbf{S}_n$. Denote also $G_n = G$ for all n. Then $(G, \mathbf{S}^{\infty})$ is a Lévy transformation group.

4. CONCENTRATION PROPERTY AND FIXED POINTS

The following definition is an attempt to capture 'concentration in the absence of measure' (as indeed there are typically no invariant measures on infinite dimensional spaces).

DEFINITION 7 [M2,M3]. Let a group G act on a metric space X by uniform isomorphisms. Call a subset $A \subseteq X$ *essential* if for every $\varepsilon > 0$ and every finite collection $g_1, \ldots, g_N \in G$ one has

$$\bigcap_{i=1}^{N} g_i A_\varepsilon \neq \varnothing .$$

(Have another look at Fig. 1 !)

EXERCISE 5. The definition obtained by replacing $g_i A_\varepsilon$ with $(g_i A)_\varepsilon$ is equivalent.

Informally speaking, an essential set is so 'big' that translates of any ε-neighbourhood of it, taken in any finite number, don't fit in without overlapping.

DEFINITION 8 (*ibidem*). A G-space X has the *concentration property* if every finite cover of X contains at least one essential set.

Perhaps one gets a better idea of the property if we start with an example where it is violated.

EXAMPLE 12 (Imre Leader, 1988, unpublished). The $U(\mathcal{H})$-space $\mathbf{S}^\infty$ (the unit sphere in $\mathcal{H} = \ell_2$) does not have the concentration property. Denote by E the set of all even natural numbers, and let P_E be the corresponding projection in ℓ_2. Set

$$A = \left\{ x \in \mathbf{S}^\infty : \|P_E x\| \geq \sqrt{2}/2 \right\},$$

$$B = \left\{ x \in \mathbf{S}^\infty : \|P_E x\| \leq \sqrt{2}/2 \right\}.$$

Clearly, $A \cup B = \mathbf{S}^\infty$. At the same time, both A and B are inessential. Indeed, let E_1, E_2, E_3 be three arbitrary disjoint infinite subsets of $\mathbf{N}$, and let $\varphi_i : \mathbf{N} \to \mathbf{N}$ be bijections with $\varphi_i(E) = E_i$, $i = 1, 2, 3$. Let g_i denote the unitary operator on $\ell_2(\mathbf{N})$ induced by φ_i. Now

$$g_i(A) = \left\{ x \in \mathbf{S}^\infty : \|P_{E_i} x\| \geq \sqrt{2}/2 \right\},$$

and consequently

$$(g_i(A))_\varepsilon \subseteq \left\{ x \in \mathbf{S}^\infty : \|P_{E_i} x\| \geq (\sqrt{2}/2) - \varepsilon \right\}.$$

Thus, as long as $\varepsilon < \sqrt{2}/2 - \sqrt{3}/3$, we have

$$\bigcap_{i=1}^{3} (g_i(A))_\varepsilon = \varnothing.$$

The set B is treated similarly.

THEOREM 2. *A compact G-space K has the concentration property if and only if it contains a fixed point: $g \cdot \kappa = \kappa$ for all $g \in G$.*

Proof. ($\Rightarrow$) CLAIM 1. *There is a point $\kappa \in K$ such that every neighbourhood of κ is essential.*

Assuming the contrary, we could have covered K with inessential open sets and, selecting a finite open subcover, obtain a contradiction.

CLAIM 2. *Any point κ as above is G-fixed.*

Again, assume that for some $g \in G$, $g \cdot \kappa \neq \kappa$. Set $\varepsilon = d(\kappa, g \cdot \kappa)/2$. Choose a number $\delta > 0$ so small that $\delta \leq \varepsilon/2$ and the g-translate of the open ball $B_\delta(\kappa)$ is contained in the $(\varepsilon/2)$-ball around $g \cdot \kappa$. The set $V = B_\delta(\kappa)$ is essential, yet the δ-neighbourhoods of V and $g \cdot V$ don't meet, a contradiction.

($\Leftarrow$) Obvious.   □

The following result provides nontrivial examples of G-spaces with the concentration property.

THEOREM 3. *Every Lévy G-space (G, X) has the concentration property.*

Proof. Let

$$\gamma = \{A_1, A_2, \ldots, A_k\}$$

be a finite cover of X. Since for each $n = 1, 2, \ldots$ the values $\mu_n(A_i)$, $i = 1, 2, \ldots, k$, add up to one, at least one of the sets in γ, let us denote it simply $A = A_i$, has the property:

$$\limsup_{n \to \infty} \mu_n(A) \geq \frac{1}{k}.$$

Now let $\varepsilon > 0$ and a finite collection $g_j, j = 1, 2, \ldots, m$ be given. Using Exercise 2, choose a number n_0 so large that

$$\mu_n(B_\varepsilon) > 1 - \frac{1}{m}$$

whenever $n > n_0$ and $\mu_n(B) \geq \frac{1}{k}$. Choose an $n > n_0$ with $\mu_n(A) \geq \frac{1}{k}$; then $\mu_n(g_j A) \geq \frac{1}{k}$ as well, and

$$\mu_n(g_j A)_\varepsilon > 1 - \frac{1}{m}, \quad i = 1, 2, \ldots, m,$$

implying that the ε-neighbourhoods of all the translates of A_ε by g_j's have a common point. $\square$

To extract useful information from the above, it only remains to link the concentration property of a G-space to that of its compactification.

LEMMA 1. *Let X and Y be two G-spaces[3]). Let $i \colon X \to Y$ be an equivariant map. If (G, X) has the concentration property, then so does (G, Y).*

Proof. If $A \subseteq X$ is an essential subset, then so is $i(A)$. Notice that the uniform continuity of i is used here in a substantial way. $\square$

The following is now immediate.

THEOREM 4 [Gr-M1]. *Let (G, X) be a Lévy G-space and let K be a compact G-space, such that there is an equivariant map $X \to K$. Then K has a G-fixed point.* $\square$

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[3]) As before, X and Y are metric spaces upon which G acts continuously, by uniform isomorphisms.

Using Theorem 1 and Example 10, we obtain

COROLLARY 1. *Whenever the topological group* $U(\ell_2)_s$ *acts continuously on a compact space, it has a fixed point.*

Such topological groups are said to have the *fixed point on compacta property,* or else to be *extremely amenable.* And indeed, this property is a drastically strengthened form of the usual amenability, which can be reformulated as follows (Day): a topological group G is amenable if and only if every affine continuous action of G on a convex compact set [in a locally convex space] has a fixed point.

REMARK 5. No locally compact group can have the fixed point on compacta property, this is a theorem by Veech ([Ve], Th. 2.2.1).

REMARK 6. The unitary group $U(\mathcal{H})_s$ was the first 'natural' extremely amenable group to be discovered. The second such discovery was the group $L_0((0,1),\mathbf{T})$ of all (equivalence classes of) measurable maps from the unit interval to the circle rotation group, equipped with the topology of convergence in measure. This was proved by Glasner (and published years later [Gl]) and, independently, by Furstenberg and Weiss (never published). This group is a Lévy group, and the approximating Lévy family of subgroups is formed by tori $\mathbf{T}^n$, made up of simple functions with respect to a refining sequence of measurable partitions of $(0,1)$.

It is interesting that both groups mentioned in the previous paragraph appear as the 'outermost' cases of a newly discovered class of extremely amenable groups. Recall that a von Neumann algebra M is *approximately finite-dimensional* if it contains a directed family of finite-dimensional $*$-subalgebras with everywhere dense union. Denote by M_* the predual of M. It is proved in [G-P] that a von Neumann algebra M is approximately finite-dimensional if and only if the unitary group of M, equipped with the topology $\sigma(M, M_*)$, is the product of a compact group with an extremely amenable group.

The two cases to consider now are $M = \mathcal{B}(\mathcal{H})$, where the unitary group with the above topology is $U(\mathcal{H})_s$, and $M = L^\infty(0,1)$, in which case the unitary group is $L_0((0,1),\mathbf{T})$.

As a corollary, nuclear C^*-algebras admit a characterization in terms of topological dynamics of their unitary groups. Recall that an action of a group G on a compact space X is *minimal* if the G-orbit of every point of X is everywhere dense, and *equicontinuous* if the family of all mappings

$x \mapsto gx$, $g \in G$ of X to itself is uniformly equicontinuous. By considering the enveloping von Neumann algebra, one can deduce that a C^*-algebra A is nuclear if and only if every minimal continuous action of the unitary group U(A), equipped with the $\sigma(A, A^*)$-topology, on a compact space K is equicontinuous.

REMARK 7. One has to be careful while applying Theorem 4. For instance, consider the infinite permutation group S_∞, formed by all self-bijections of a countably infinite set, say $\mathbf{Z}$. This group is equipped with the natural Polish topology of pointwise convergence on discrete $\mathbf{Z}$, induced by the embedding $S_\infty \hookrightarrow \mathbf{Z}^{\mathbf{Z}}$. The idea of applying concentration in finite groups of permutations (Example 3) to conclude that S_∞ is a Lévy group is attractive, but does not work.

EXERCISE 6. Let d be any right-invariant metric on S_∞, generating the topology of pointwise convergence. Show that S_∞, acting on the left upon (S_∞, d), does not have the concentration property.

[*Hint.* Let τ be the transposition exchanging 0 and 1 and leaving the rest of $\mathbf{Z}$ fixed. Choose $\varepsilon > 0$ so that the ε-ball around e_G is contained in the intersection of the isotropy subgroups of 0 and 1. Now partition S_∞ into two sets A and B, where

$$A = \{\sigma \in S_\infty : \sigma^{-1}(0) < \sigma^{-1}(1)\}$$

and $B = S_\infty \setminus A$. Try to apply the concentration property to the cover $\{A, B\}$, the number ε, and the collection of two elements e, τ.]

It follows that S_∞ acts on some compact space without fixed points. (This was noted in [P1].) Very recently such an action was constructed explicitly by Eli Glasner and Benji Weiss [Gl-W]. We will return to their construction later (Subsection 6.4).

One can even show that S_∞ is not a Lévy group no matter what the group topology is ([P2], Remark 4.9). However, it is still possible to put the finite permutation groups (S_n) together so as to obtain a Lévy group.

This is the group Aut(X, μ) of all measure-preserving automorphisms of the standard non-atomic Lebesgue space, (X, μ), equipped with the weak topology, that is, the weakest topology making every map of the form Aut(X, μ) $\ni \tau \mapsto \mu(A \triangle \tau(A)) \in \mathbf{R}$ continuous, where $A \subseteq X$ is a measurable set. This group contains finite permutation groups, realized as subgroups of interval exchange transformations, and any right-invariant metric makes those

subgroups into a Lévy family. A similar result holds for the group $\text{Aut}^*(X, \mu)$ of all measure class preserving transformations (Thierry Giordano and the author [G-P]).

5. INVARIANT MEANS ON SPHERES

Let a group G act on a metric space X by uniform isomorphisms. The formula

$$^g f(x) = f(g^{-1} \cdot x)$$

determines an action of G on the space $\text{UCB}(X)$ of all uniformly continuous bounded complex valued functions on X by linear isometries. If G is a topological group acting on X continuously, the above action of G on $\text{UCB}(X)$ need not, in general, be continuous. (An example: $G = \text{U}(\ell_2)_s$, $X = \mathbf{S}^\infty$.) However, the action will be continuous if X is compact. (An easy check.) To some extent, the latter observation can be inverted.

EXERCISE 7. Let a topological group G act continuously on a commutative unital C^*-algebra A by automorphisms. Then this action determines a continuous action of G on the space of maximal ideals of A, equipped with the usual (weak*) topology.

Recall that a *mean* on a space $\mathcal{F}$ of functions is a positive linear functional, m, of norm one, sending the function 1 to 1. A mean is *multiplicative* if $\mathcal{F}$ is an algebra and the mean is a homomorphism of this algebra to $\mathbf{C}$.

COROLLARY 2. *Let (G, X) be a Lévy G-space. Then there exists a G-invariant multiplicative mean on the space $\text{UCB}(X)$ of all bounded uniformly continuous functions on X.*

Proof. According to Exercise 7, the group G acts continuously on the space $\mathfrak{M}$ of maximal ideals of the C^*-algebra $\text{UCB}(X)$. Therefore, $\mathfrak{M}$ is an equivariant compactification of X. By Theorem 4, there is a fixed point $\varphi \in \mathfrak{M}$, which is the desired invariant multiplicative mean. $\quad \square$

The following is deduced by considering Example 11.

COROLLARY 3 [Gr-M1]. *If a compact group G is represented by unitary operators in an infinite-dimensional Hilbert space $\mathcal{H}$, then there exists a G-invariant multiplicative mean on the uniformly continuous bounded functions on the unit sphere of $\mathcal{H}$.*

REMARK 8. The infinite-dimensionality of $\mathcal{H}$ is essential. Since the unit sphere $\mathbf{S}$ of a finite-dimensional space $\mathcal{H}$ is compact, an invariant multiplicative mean on UCB($\mathbf{S}$) exists if and only if there is a fixed vector $\xi \in \mathbf{S}$.

Means on UCB(X), where $X = \mathbf{S}^\infty$ is the unit sphere in the Hilbert space, as well as some other infinite-dimensional manifolds, were studied by Paul Lévy, who viewed them as (substitutes for) infinite-dimensional integrals[4]. The invariant means can thus serve as a substitute for invariant integration on the infinite-dimensional spheres. One can substantially generalize Corollary 3. With this purpose in view, it is convenient to enlarge the concept of a Lévy transformation group.

If μ_1, μ_2 are probability measures on the same metric space X, then the *transportation distance* between them is defined as

$$d_{tran}(\mu_1, \mu_2) = \inf \int_{X \times X} d(x, y)\, d\nu(x, y),$$

where the infimum is taken over all probability measures ν on the product space $X \times X$ such that $(\pi_i)_* \nu = \mu_i$ for $i = 1, 2$ and $\pi_1, \pi_2 \colon X \times X \to X$ denote the coordinate projections.

The way to think of the transportation distance is to identify each probability measure with a pile of sand, then $d_{tran}(\mu_1, \mu_2)$ is the minimal average distance that each grain of sand has to travel when the first pile is being moved to take the place of the second[5].

Let us from now on replace Definition 6 with the following, more general one.

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[4]) The multiplicativity of some of those means, which is not exactly a property one expects of an integral, becomes clear if one recalls an equivalent way to express the concentration phenomenon: on a high-dimensional structure, every 1-Lipschitz function is, probabilistically, almost constant, cf. Section 7.

[5]) In computer science, the transportation distance is known as the Earth Mover's Distance (EMD).

DEFINITION 9. Say that a G-space (G, X) is Lévy if there is a net of probability measures (μ_α) on X, such that the mm-spaces (X, d, μ_α) form a Lévy family and for each $g \in G$,

$$d_{tran}(\mu_\alpha, g\mu_\alpha) \to 0.$$

Theorems 3 and 4 remain true, with very minor modifications of the proofs.

Here is one application. A unitary representation π of a group G in a Hilbert space $\mathcal{H}$ is *amenable* in the sense of Bekka [Be] if there exists a state, φ, on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the space $\mathcal{H}$ of representation, which is invariant under the action of G by inner automorphisms: $\varphi(\pi_g T \pi_g^*) = \varphi(T)$ for every $T \in B(\mathcal{H})$ and every $g \in G$.

THEOREM 5 [P2]. *Let π be a unitary representation of a group G in a Hilbert space $\mathcal{H}$. The following are equivalent.*

(i) π *is amenable.*

(ii) *Either π has a finite-dimensional subrepresentation, or $(G, \mathbf{S})$ has the concentration property (or both).*

(iii) *There is a G-invariant mean on the space* UCB($\mathbf{S}$) *(a 'Lévy-type integral').*

Proof. (i) $\Rightarrow$ (ii): according to Th. 6.2 and Remark 1.2.(iv) in [Be], a representation π is amenable if and only if for every finite set $g_1, g_2, \ldots, g_k$ of elements of G and every $\varepsilon > 0$ there is a projection P of finite rank such that for all $i = 1, 2, \ldots, k$

$$\left\| P - \pi_{g_i} P \pi_{g_i}^* \right\|_1 < \varepsilon \|P\|_1,$$

where $\| \cdot \|_1$ denotes the trace class operator norm. It follows that the transportation distance between the Haar measure on the unit sphere in the range of the projection P and the translates of this measure by operators π_{g_i} can be made as small as desired via a suitable choice of P. Now a variant of Theorem 4 applies. (See [P2] for details.)

(ii) $\Rightarrow$ (iii): in the first case, the mean is obtained by invariant integration on the finite-dimensional sphere, while in the second case even a multiplicative mean exists.

(iii) $\Rightarrow$ (i): let ψ be a G-invariant mean on UCB($\mathbf{S}_\mathcal{H}$). For every bounded linear operator T on $\mathcal{H}$ define a (Lipschitz) function $f_T : \mathbf{S}_\mathcal{H} \to \mathbf{C}$ by

$$\mathbf{S}_\mathcal{H} \ni \xi \mapsto f_T(\xi) := \langle T\xi, \xi \rangle \in \mathbf{C},$$

and set $\varphi(T) := \psi(f_T)$. This φ is a G-invariant mean on $\mathcal{B}(\mathcal{H})$. $\square$

COROLLARY 4. *A locally compact group G is amenable if and only if for every strongly continuous unitary representation of G in an infinite-dimensional Hilbert space the pair $(G, \mathbf{S}^\infty)$ has the property of concentration.*

COROLLARY 5. *There is no invariant mean on $\mathrm{UCB}(\mathbf{S}^\infty)$ for the full unitary group $\mathrm{U}(\ell_2)$.*

Proof. If such a mean existed, then every unitary representation of every group would be amenable, in particular every group would be amenable (by Th. 2.2 in [Be]). $\square$

(Of course Corollary 5 also follows from Imre Leader's Example 12 modulo Theorem 2 and Lemma 1.)

A (not necessarily locally compact) topological group G is *amenable* if there is a left-invariant mean on the space $\mathrm{RUCB}(G)$ of all right uniformly continuous bounded functions on G. Denote by $\mathrm{U}(\ell_2)_u$ the full unitary group with the uniform operator topology.

COROLLARY 6 (Pierre de la Harpe [dlH], proved by different means). *The topological group $\mathrm{U}(\ell_2)_u$ is not amenable.*

Proof. Choose an arbitrary $\xi \in \mathbf{S}^\infty$. To every function $\psi \in \mathrm{UCB}(\mathbf{S}^\infty)$ associate the function $\widetilde{\psi}$ as follows:

$$G \ni g \mapsto \widetilde{\psi}(g) := \psi(\pi_g^*(\xi)) \in \mathbf{C}.$$

The correspondence $\psi \mapsto \widetilde{\psi}$ is a G-equivariant positive bounded unit-preserving linear operator from $\mathrm{UCB}(\mathbf{S}^\infty)$ to $\mathrm{RUCB}(\mathrm{U}(\ell_2)_u)$, and any left-invariant mean φ on the latter G-module would thus determine a G-invariant mean on the former G-module, contradicting Corollary 5. $\square$

EXAMPLE 13. In a similar fashion, by considering the action of $\mathrm{Aut}(X, \mu)$ on $L_0^2(X, \mu)$, where $X = \mathrm{SL}(3, \mathbf{R})/\mathrm{SL}(3, \mathbf{Z})$, one deduces that $\mathrm{Aut}(X, \mu)_u$ with the uniform topology is not amenable [G-P].

6. Ramsey–Dvoretzky–Milman property

6.1 Extreme amenability and small oscillations

One way to intuitively describe a 'Ramsey-type result' is as follows. Suppose $\mathfrak{X}$ is a large (and often highly homogeneous) structure of some sort or other. Let $\mathfrak{X}$ be partitioned into a finite number of pieces in an arbitrary way. No matter how irregular and 'ragged' the pieces are, at least one of them always contains the remnants of the original structure, that is, a (possibly much smaller, but still detectable) substructure of the same type which survived intact.

We are now going to link explicitly the fixed point on compacta property to Ramsey-type results. Here is the first step.

EXERCISE 8. Prove that a topological group G is extremely amenable if and only if for every finite collection $g_1, \ldots, g_n$ of elements of G, every bounded right uniformly continuous function $f \colon G \to \mathbf{R}^N$ from G to a finite-dimensional Euclidean space, and every $\varepsilon > 0$ there is an $h \in G$ such that $|f(h) - f(g_i h)| < \varepsilon$ for each $i = 1, 2, \ldots, n$.

[*Hints.* ($\Rightarrow$) The action of G on the space $\mathcal{S}(G)$ of maximal ideals of the C^*-algebra RUCB(G) is continuous, and G itself can be thought of as an everywhere dense subset of $\mathcal{S}(G)$.

($\Leftarrow$) Form a net of suitably indexed elements h as above and consider any limit point of the net $h_\alpha \cdot \xi$, where ξ is an arbitrary element of the compact space upon which G acts continuously.]

EXERCISE 9. Prove that the above condition for extreme amenability is, in turn, equivalent to the following. For every bounded *left* uniformly continuous function f from G to a finite-dimensional Euclidean space, every finite subset F of G, and every $\varepsilon > 0$, the oscillation of f on a suitable left translate of F is less than ε :

$$\exists g \in G, \ \mathrm{Osc}(f|_{gF}) < \varepsilon.$$

It is convenient to deal with the above property in a more general context of G-spaces.

DEFINITION 11 [Gr1]. Say that a G-space X (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$, and every finite $F \subseteq X$, there is a $g \in G$ with the property

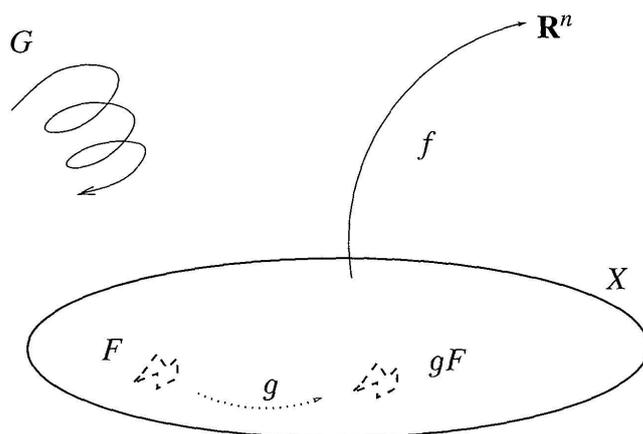$$\mathrm{Osc}(f|_{gF}) < \varepsilon .$$

FIGURE 3

The Ramsey–Dvoretzky–Milman property

REMARK 9. Equivalently, F can be assumed compact.

COROLLARY 7. *For a topological group G the following are equivalent:*

 (i) *G is extremely amenable;*

 (ii) *every metric space X upon which G acts continuously and transitively by isometries has the R–D–M property;*

 (iii) *every homogeneous factor-space G/H, equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.*

Next, we will discover two very important situations where the R–D–M property appears naturally.

6.2 DVORETZKY'S THEOREM

Here is the famous result.

THEOREM (Arieh Dvoretzky). *For all $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ such that for any n-dimensional normed space $(X, \|\cdot\|_E)$ there is a subspace V of $\dim V \geq c \log n$ and a Euclidean norm $\|\cdot\|_2$ with $\|x\|_2 \leq \|x\|_E \leq (1 + \varepsilon)\|x\|_2$ for all $x \in V$.*

The studies of the phenomenon of concentration of measure were given a boost by Vitali Milman's new proof of the Dvoretzky theorem [M1], based on a suitable finite-dimensional approximation to the lemma which follows directly from results that we have previously stated:

LEMMA (Milman).   *The pair* $(U(\mathcal{H}), S^\infty)$ *has the R–D–M property, where* S^∞ *is the unit sphere of an infinite-dimensional Hilbert space* $\mathcal{H}$.

6.3 RAMSEY'S THEOREM

Let r be a positive natural number. By $[r]$ one denotes the set $\{1, 2, \ldots, r\}$. A *colouring* of a set X with r colours, or simply r-*colouring*, is any map $\chi \colon X \to [r]$. A subset $A \subseteq X$ is *monochromatic* if for every $a, b \in A$ one has $\chi(a) = \chi(b)$.

Put otherwise, a finite colouring of a set X is nothing but a partition of X into finitely many (disjoint) subsets.

Let X be a set, and let k be a natural number. Denote by $[X]^k$ the set of all k-subsets of X, that is, all (unordered !) subsets containing exactly k elements.

INFINITE RAMSEY THEOREM.   *Let* X *be an infinite set, and let* k *be a natural number. For every finite colouring of* $[X]^k$ *there exists an infinite subset* $A \subseteq X$ *such that the set* $[A]^k$ *is monochromatic.*
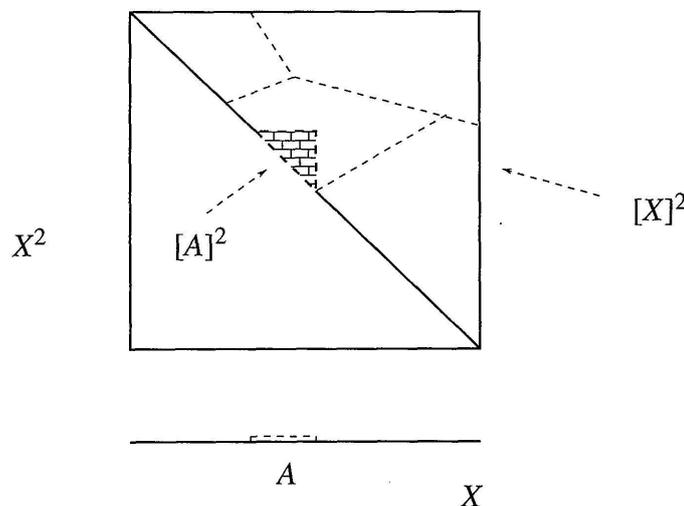

FIGURE 4

Ramsey theorem for $k = 2$

REMARK 10. For $k = 1$ the statement is simply the pigeonhole principle. Here is a popular interpretation of the result in the case $k = 2$. Among infinitely many people, either there is an infinite subset of people every two of whom know each other, or there is an infinite subset no two members of which know each other.

FINITE RAMSEY THEOREM. *For every triple of natural numbers, k, l, r, there exists a natural number $R(k, l, r)$ with the following property. If $N \geq R(k, l, r)$ and the set of all k-subsets of $[N]$ is coloured using r colours, then there is a subset $A \subseteq [N]$ of cardinality $|A| = l$ such that all k-subsets of A have the same colour.*

REMARK 11. The Infinite Ramsey Theorem implies the finite version through a simple compactness argument. At the same time, the infinite version does not seem quite to follow from the finite one. The finite version is equivalent to the following statement:

Let X be an infinite set, and let k be a natural number. For every finite colouring of $[X]^k$ and every natural n there exists a subset $A \subseteq X$ of cardinality n such that $[A]^k$ is monochromatic.

A good introductory reference to Ramsey theory is [Gra].

Denote by $\mathrm{Aut}(\mathbf{Q})$ the group of all order-preserving bijections of the set of rational numbers, equipped with the topology of pointwise convergence on the discrete set $\mathbf{Q}$. In other words, we regard $\mathrm{Aut}(\mathbf{Q})$ as a (closed) topological subgroup of S_∞. A basic system of neighbourhoods of identity is formed by open subgroups each of which stabilizes elements of a given finite subset of $\mathbf{Q}$.

EXERCISE 10. Use Corollary 7 to prove that the finite Ramsey theorem is equivalent to the statement:

The topological group $\mathrm{Aut}(\mathbf{Q})$ *is extremely amenable.*

[*Hint.* For a finite subset $M \subset \mathbf{Q}$, the left factor space of $\mathrm{Aut}(\mathbf{Q})$ by the stabilizer of M can be identified with the set $[\mathbf{Q}]^n$, where $n = |M|$, equipped with the discrete uniformity (or $\{0, 1\}$-valued metric). Cover $[\mathbf{Q}]^n$

with finitely many sets on each of which the given function f has oscillation $< \varepsilon$, and apply Ramsey's theorem. Use Remark 11.]

6.4 EXTREME AMENABILITY AND MINIMAL FLOWS

COROLLARY 8. *The group of orientation-preserving homeomorphisms of the closed unit interval,* $\mathrm{Homeo}_+(\mathbf{I})$, *equipped with the compact-open topology, is extremely amenable.*

Proof. Indeed, the extremely amenable group $\mathrm{Aut}(\mathbf{Q})$ admits a continuous monomorphism with a dense image into the group $\mathrm{Homeo}_+(\mathbf{I})$.

REMARK 12. Thompson's group F consists of all piecewise-linear homeomorphisms of the interval whose points of non-smoothness are finitely many dyadic rational numbers, and the slopes of any linear part are powers of 2. (See [CFP].) It is a major open question in combinatorial group theory whether the Thompson group is amenable. Since F is everywhere dense in $\mathrm{Homeo}_+(\mathbf{I})$, our Corollary 8 does not contradict the possible amenability of F.

Using the extreme amenability of the topological groups $\mathrm{Aut}(\mathbf{Q})$ and $\mathrm{Homeo}_+(\mathbf{I})$, one is able to compute explicitly the universal minimal flows of some larger topological groups as follows.

COROLLARY 9. *The circle* $\mathbf{S}^1$ *forms the universal minimal* $\mathrm{Homeo}_+(\mathbf{S}^1)$-*space.*

Proof. Let $\theta \in \mathbf{S}^1$ be an arbitrary point. The isotropy subgroup St_θ of θ is isomorphic to $\mathrm{Homeo}_+(\mathbf{I})$. Because of that, whenever the topological group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts continuously on a compact space X, the subgroup St_θ has a fixed point, say $x' \in X$. The mapping $\mathrm{Homeo}_+(\mathbf{S}^1) \ni h \mapsto h(x') \in X$ is constant on the left St_θ-cosets and therefore gives rise to a continuous equivariant map $\mathrm{Homeo}_+(\mathbf{S}^1)/\mathrm{St}_\theta \cong \mathbf{S}^1 \to X$.

For the above results concerning groups $\mathrm{Aut}(\mathbf{Q})$, $\mathrm{Homeo}_+(\mathbf{I})$, and $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [P1].

Now denote by LO the set of all linear orders on $\mathbf{Z}$, equipped with the (compact) topology induced from $\{0,1\}^{\mathbf{Z} \times \mathbf{Z}}$. The group S_∞ acts on LO by double permutations.

EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order $\prec$ is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η. The collection LO_η of all linear orders of type η on $\mathbf{Z}$ can be identified with the factor space $S_\infty / \mathrm{Aut}(\prec)$ through the correspondence $\sigma \mapsto {}^\sigma\prec$. Here $\prec$ is some chosen linear order of type η on $\mathbf{Z}$ and $\mathrm{Aut}(\prec)$ stands for the group of order-preserving self-bijections of $(\mathbf{Z}, \prec)$, acting on the space of orders in a natural way: $(x \,{}^\sigma\!\prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η, induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \to S_\infty / \mathrm{Aut}(\prec) \cong LO_\eta$ right uniformly continuous.

Let now X be a compact S_∞-space. The topological subgroup $\mathrm{Aut}(\prec)$ of S_∞ has a fixed point in X, say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\mathrm{Aut}(\prec)$-cosets and thus gives rise to a mapping $\varphi: LO_\eta \to X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞-spaces LO $\to X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space* LO *forms the universal minimal S_∞-space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* **U** [Ur] is determined uniquely (up to an isometry) by the following conditions:

  (i) **U** is a complete separable metric space;

 (ii) **U** is ω-homogeneous, that is, every isometry between two finite subspaces of **U** extends to an isometry of **U**;

(iii) **U** contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a 'sufficiently random' metric is almost surely isometric to **U**.

The group of isometries Iso (**U**) with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses

a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group $\mathrm{Iso}\,(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into $\mathbf{U}$ be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into $\mathbf{U}$ that factor through G are monochromatic to within ε.
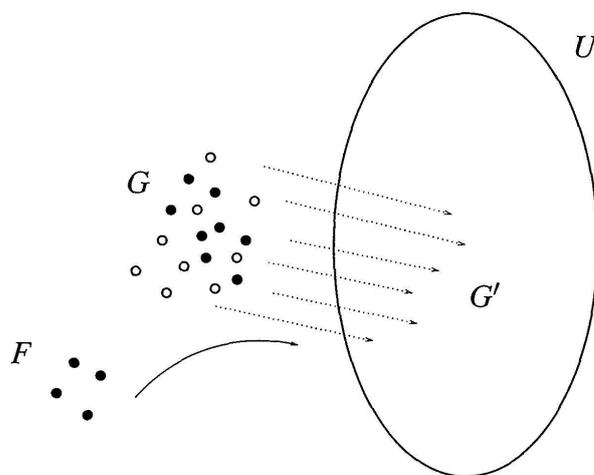
FIGURE 5

A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within* ε if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A. In our case, the Hausdorff distance is formed with regard to the uniform metric on $\mathbf{U}^F$.

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere $\mathbf{S}^\infty$ in ℓ_2 [P3].   •

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an mm-space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.

EXERCISE 14.   Assume that a function f as above is 1-Lipschitz, that is, $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$. Prove that for every $\varepsilon > 0$,

$$\mu\{|f(x) - M_f| > \varepsilon\} \leq 2\alpha_X(\varepsilon).$$

Thus, one can express the phenomenon of concentration of measure by stating that on a 'high-dimensional' mm-space, every Lipschitz (more generally, uniformly continuous) function is, probabilistically, almost constant.

Following Gromov [Gr3, $3\frac{1}{2}.45$], let us recast the concentration phenomenon yet again.

On the space $L(0, 1)$ of all measurable functions define the metric me_1, generating the topology of convergence in measure, by letting $me_1(h_1, h_2)$ stand for the infimum of all $\lambda > 0$ with the property

$$\mu^{(1)}\{|h_1(x) - h_2(x)| > \lambda\} < \lambda.$$

(Here $\mu^{(1)}$ denotes the Lebesgue measure on the unit interval $\mathbf{I} = [0, 1]$.)

Now let $X = (X, d_X, \mu_X)$ and $Y = (Y, d_Y, \mu_Y)$ be two Polish mm-spaces. There exist measurable maps $f \colon \mathbf{I} \to X$, $g \colon \mathbf{I} \to Y$ such that $\mu_X = f_* \, \mu^{(1)}$ and $\mu_Y = g_* \, \mu^{(1)}$. Denote by L_f the set of all functions of the form $h = h_1 \circ f$, where $h_1 \colon X \to \mathbf{R}$ is 1-Lipschitz, having the property $h(0) = 0$. Similarly, define the set L_g. Now define a non-negative real number $\underline{H}_1 \mathcal{L}\iota(X, Y)$ as the infimum of Hausdorff distances between L_f and L_g (formed using the metric me_1 on the space of functions), taken over all parametrizations f and g as above.
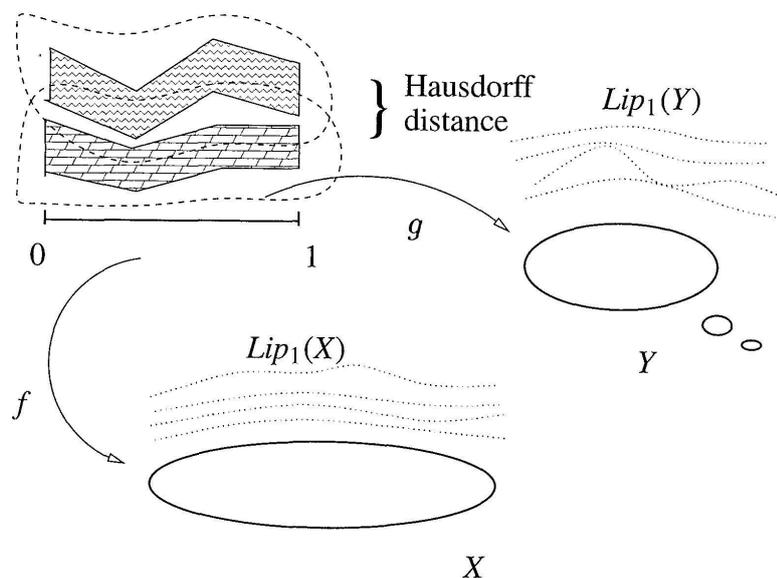

FIGURE 6

Gromov's distance $\underline{H}_1 \mathcal{L}\iota$ between mm-spaces

EXERCISE 15. Prove that $\underline{H}_1\mathcal{L}\iota$ is a metric on the space of (isomorphism classes of) all Polish mm-spaces.

EXERCISE 16. Prove that a sequence of mm-spaces $X_n = (X_n, d_n, \mu_n)$ forms a Lévy family if and only if it converges to the trivial mm-space in the metric $\underline{H}_1\mathcal{L}\iota$:

$$X_n \xrightarrow{\underline{H}_1\mathcal{L}\iota} \{*\}.$$

If one now replaces the trivial space on the right hand side with an arbitrary mm-space [6]), one obtains the concept of *concentration to a non-trivial space*.

According to Gromov, this type of concentration commonly occurs in statistical physics. At the same time, there are very few known non-trivial examples of this kind in the context of transformation groups.

Here is just one problem in this direction, suggested by Gromov. Every probability measure ν on a group G determines a random walk on G. How can one associate to (G, ν) in a natural way a sequence of mm-spaces which would concentrate to the boundary [Fur] of the random walk?

8. READING SUGGESTIONS

The 2001 Borel seminar was based on Chapter $3\frac{1}{2}$ of the green book [Gr3], which contains a wealth of ideas and concepts and can be complemented by [Gr4]. The survey [M3] by Vitali Milman, to whom we owe the present status of the concentration of measure phenomenon, is highly relevant and rich in material, especially if read in conjunction with a recent account of the subject by the same author [M4]. The book [M-S] is, in a sense, indispensable and should always be within one's reach. Talagrand's fundamental paper [Ta1] has to be at least browsed by every learner of the subject, while the paper [Ta2] of the same author offers an independent introduction to the subject of concentration of measure. The newly-published book by Ledoux [Led], apparently the first ever monograph devoted exclusively to concentration, is highly readable and covers a wide range of topics. Don't miss the introductory survey by Schechtman [Sch]. The modern setting for concentration was designed in the important paper [Gr-M1] by Gromov and Milman, which had also introduced the subject of this lecture and from which many results (perhaps with slight modifications) have been taken.

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[6]) Or, more generally, a uniform space — for instance, a non-metrizable compact space — with measure.

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