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Thus, the case of a free nilpotent Lie algebra  $f_c(p, \mathbf{C})$  (as a Lie algebra over  $\mathbf{R}$ ) on  $p$  generators differs from the case 2.2.

REMARK. All rational forms of  $f_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$  and  $f_2(2, \mathbf{R}) \oplus f_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$  are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type  $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$  where  $F_c(p, \mathbf{R})$  is the free nilpotent Lie group on  $p$  free generators.*

#### 4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let  $m$  be a rational number and  $A_m = \mathbf{Q}[x]/(x^2 - m)$ .  $A_m$  is a 2-dimensional commutative algebra over  $\mathbf{Q}$  which depends only on  $m$  modulo square factors. Thus there are four types of  $A_m$ :

- 1) if  $m = 1$  then  $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$ ;
- 2) if  $m > 1$  is a positive square-free integer then  $A_m \cong \mathbf{Q}(\sqrt{m})$  is a real quadratic field over  $\mathbf{Q}$ ;
- 3) if  $m = 0$  then  $A_0$  is the algebra of dual numbers over  $\mathbf{Q}$ ;
- 4) if  $m$  is a negative square-free integer then  $A_m \cong \mathbf{Q}(\sqrt{m})$  is an imaginary quadratic field over  $\mathbf{Q}$ .

Let  $\mathfrak{hei}_3(A_m)$  be a Heisenberg algebra over  $A_m$  considered over  $\mathbf{Q}$ . Then  $\mathfrak{hei}_3(A_m)$  is a rational form of either  $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ , or  $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ , or  $\mathfrak{hei}_3(\mathbf{C})$ . More precisely,

THEOREM 3. *Let  $\mathfrak{h}$  be a 6-dimensional nilpotent Lie algebra of class 2 over  $\mathbf{Q}$ . Suppose that  $[\mathfrak{h}, \mathfrak{h}]$  coincides with the 2-dimensional centre of  $\mathfrak{h}$ . Then  $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$  for some  $m \in \mathbf{Q}$  as above.*

Moreover,

- 1)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$  iff  $m > 0$ ,
- 2)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$  iff  $m = 0$ ,
- 3)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$  iff  $m < 0$ ,

and up to isomorphism there are no more rational forms for  $\mathfrak{g}_-$ ,  $\mathfrak{g}_0$ ,  $\mathfrak{g}_+$ . The Lie algebras  $\mathfrak{hei}_3(A_m)$  and  $\mathfrak{hei}_3(A_n)$  are isomorphic over  $\mathbf{Q}$  if and only if  $A_m$  and  $A_n$  are isomorphic.

*Proof.* Take some  $\mathbf{Q}$ -basis  $x_1, \dots, x_6$  of  $\mathfrak{h}$ . First of all, we may suppose that  $[x_1, x_2] = x_5$  (possibly after a change of basis). Thus  $x_5$  is central. We have to deal with two cases.

CASE 1. All brackets  $[x_1, x_j]$ ,  $[x_2, x_j]$  ( $j \geq 3$ ) are multiples of  $x_5$ . If  $[x_1, x_j] = a_j x_5$ ,  $[x_2, x_j] = b_j x_5$  then we set  $X_j = x_j - a_j x_2 + b_j x_1$  whence  $[x_1, X_j] = [x_2, X_j] = 0$ .

Since  $[\mathfrak{h}, \mathfrak{h}]$  is 2-dimensional we conclude that some commutator, say  $[x_3, x_4]$ , is not a multiple of  $x_5$  (for convenience, we use lower-case 'x' instead of 'X'). Consider

$$(4.1) \quad [x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

Commuting  $[x_3, x_4]$  with  $x_1, x_2$  we obtain that  $a = b = 0$ . Let us suppose that  $f = 0$ . Then

$$(4.2) \quad [x_3, x_4] = cx_3 + dx_4 + ex_5.$$

Recall that  $x_5$  and  $[x_3, x_4]$  in the form (4.2) span the 2-dimensional centre. Commuting  $cx_3 + dx_4 + ex_5$  from (4.2) with  $x_3, x_4$  we get  $c = d = 0$  and a contradiction. Thus  $f \neq 0$ . We may assume that  $[x_3, x_4] = x_6$  where  $x_6$  is central. Hence, we have the following multiplication table for  $\mathfrak{h}$ :  $[x_1, x_2] = x_5$ ,  $[x_3, x_4] = x_6$ , other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}).$$

CASE 2. Among the brackets  $[x_1, x_j]$ ,  $[x_2, x_j]$  ( $j \geq 3$ ) there is at least one which is not a multiple of  $x_5$ . In this case we may suppose (changing indices if necessary) that this bracket is  $[x_1, x_3]$ . Let

$$(4.3) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6 \quad .$$

and let us suppose that  $d = f = 0$ . Then

$$(4.4) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5.$$

Commuting the right-hand term of (4.4) with  $x_1$  we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2x_3 + (ce + b)x_5.$$

Hence  $c = b = 0$ . By virtue of this  $a = 0$  and we obtain a contradiction if we commute both sides of (4.4) with  $x_2$ . It follows that either  $d \neq 0$  or  $f \neq 0$ . In other words, we may suppose that  $[x_1, x_3]$  is equal to  $x_6$ .

Now

$$(4.5) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6$$

where  $x_5, x_6$  span  $[\mathfrak{h}, \mathfrak{h}]$ . Suppose that  $[x_2, x_3] = ax_5 + bx_6$ . Adding if necessary some multiples of  $x_1$  to  $x_2$  and  $x_3$  we obtain  $[x_2, x_3] = 0$ . In the same way we may suppose that  $[x_1, x_4] = 0$ . Adding to  $x_4$  some multiple of  $x_1$  we also obtain a relation  $[x_2, x_4] = Cx_6$ . Moreover, after scaling  $x_4$  we get  $C = 0$  or  $C = 1$ . Thus,  $\mathfrak{h}$  has a basis in which the *non-trivial* brackets are the following:

$$(4.6) \quad \begin{aligned} [x_1, x_2] &= x_5, & [x_1, x_3] &= x_6, \\ [x_2, x_4] &= Cx_6 \quad (C = 0 \text{ or } C = 1), & [x_3, x_4] &= Ax_5 + Bx_6. \end{aligned}$$

In any case  $A^2 + B^2 + C^2 \neq 0$  because  $x_4$  cannot belong to the 2-dimensional centre of  $\mathfrak{h}$ .

We will show that we can always make  $C = 1$  and  $B = 0$  in (4.6).

SUBCASE 2.1. If  $C = 0$  then the following basis transformation

$$(4.7) \quad \begin{aligned} X_1 &= x_1, & X_2 &= ax_2 + x_3, \\ X_3 &= Ax_2 + Bx_3, & X_4 &= x_4, \end{aligned}$$

yields ( $a$  is any constant such that  $aB \neq A$ )

$$(4.8) \quad \begin{aligned} [X_1, X_2] &= ax_5 + x_6 = X_5, & [X_1, X_3] &= Ax_5 + Bx_6 = X_6, \\ [X_2, X_4] &= Ax_5 + Bx_6 = X_6, & [X_3, X_4] &= B(Ax_5 + Bx_6) = BX_6. \end{aligned}$$

From now on we may suppose that  $C = 1$  in (4.6) and we arrive at

SUBCASE 2.2:  $C = 1, A = 0$ . Let

$$(4.9) \quad \begin{aligned} X_1 &= x_1 + ax_4, & X_2 &= x_2 - ax_3, \\ X_3 &= x_2 + dx_3, & X_4 &= -x_1 + dx_4, \end{aligned}$$

where  $a, d, a + d \neq 0, aB \neq 1, dB \neq -1$ . Hence

$$(4.10) \quad \begin{aligned} [X_1, X_2] &= x_5 + (a^2B - 2a)x_6 = X_5, \\ [X_1, X_3] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_2, X_4] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_3, X_4] &= x_5 + (d^2B + 2d)x_6 = \lambda X_5 + (1 - \lambda)X_6. \end{aligned}$$

Since  $a, d$  and  $a + d$  are all non-zero,  $X_5$  and  $X_6$  are linearly independent. Straightforward computations yield

$$\lambda = \frac{dB + 1}{aB - 1} \neq 0, 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1:  $C = 1$ ;  $A, B, 4A + B^2 \neq 0$ . Let now

$$(4.11) \quad \begin{aligned} X_1 &= x_1 + tx_4, & X_2 &= x_2 - tx_3, \\ X_3 &= x_3, & X_4 &= x_4, \end{aligned}$$

where  $t = -B/2A$ . Hence

$$(4.12) \quad \begin{aligned} [X_1, X_2] &= (1 + t^2A)x_5 + (t^2B - 2t)x_6 = X_5, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = \alpha X_5 = \frac{4A^2}{4A+B^2} X_5. \end{aligned}$$

SUBCASE 2.3.2:  $C = 1$ ;  $A, B \neq 0, 4A + B^2 = 0$ . The same transformation (4.11) yields

$$(4.13) \quad \begin{aligned} [X_1, X_2] &= 0, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = X_5 \end{aligned}$$

and, after the transformation  $x_1 = X_3, x_2 = X_4, x_3 = X_1, x_4 = X_2, x_5 = X_5, x_6 = -X_6$ , we obtain (4.12) with  $\alpha = 0$ . Anyway, we obtain the desired form of  $\mathfrak{h}$

$$(4.14) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6, \quad [x_2, x_4] = x_6, \quad [x_3, x_4] = Ax_5.$$

Scaling  $x_3, x_4$  by  $\lambda \neq 0$  we may suppose that  $A = m$  where  $m$  is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism  $\rho: \mathfrak{h} \rightarrow \mathfrak{h}e_3(A_m)$ . Recall that  $A_m$  has a basis  $1, x$  over  $\mathbf{Q}$  such that  $x^2 = m$ . Here are the matrices representing  $\rho(x_i)$  if  $m \neq 1$  (the case  $m = 1$  is left to the reader as an easy exercise):

$$(4.15) \quad \begin{aligned} \rho(x_1) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_5) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \rho(x_3) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_4) &= \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_6) &= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now it is evident that  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R}$  is isomorphic to either  $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ , or  $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ , or  $\mathfrak{hei}_3(\mathbf{C})$  depending on the sign of  $m$ . Thus, we have classified up to  $\mathbf{Q}$ -isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete.  $\square$

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over  $\mathbf{R}$ . Indeed, the centralizer of any element in  $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbf{C})$  is even dimensional over  $\mathbf{R}$  since this algebra can be viewed as a complex Lie algebra, whereas in both  $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$  and  $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$  there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer  $C(x)$  will not be changed if we scale  $x$  by any  $\lambda \neq 0$  or add to  $x$  any central element. This means that dimension of the centralizer is a well-defined function on the projective space  $\mathbf{P}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$  where  $\mathfrak{g}$  is either  $\mathfrak{g}_+$  or  $\mathfrak{g}_0$ . Straightforward computations show that in  $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0])$  all points with 5-dimensional centralizer belong to a unique line whereas in  $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+])$  the points under consideration form two disjoint lines.

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