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Thus, the case of a free nilpotent Lie algebra $\mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over \mathbf{R}) on p generators differs from the case 2.2.

REMARK. All rational forms of $\mathfrak{f}_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$ and $\mathfrak{f}_2(2, \mathbf{R}) \oplus \mathfrak{f}_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$ where $F_c(p, \mathbf{R})$ is the free nilpotent Lie group on p free generators.*

4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let m be a rational number and $A_m = \mathbf{Q}[x]/(x^2 - m)$. A_m is a 2-dimensional commutative algebra over \mathbf{Q} which depends only on m modulo square factors. Thus there are four types of A_m :

- 1) if $m = 1$ then $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$;
- 2) if $m > 1$ is a positive square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is a real quadratic field over \mathbf{Q} ;
- 3) if $m = 0$ then A_0 is the algebra of dual numbers over \mathbf{Q} ;
- 4) if m is a negative square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is an imaginary quadratic field over \mathbf{Q} .

Let $\mathfrak{hei}_3(A_m)$ be a Heisenberg algebra over A_m considered over \mathbf{Q} . Then $\mathfrak{hei}_3(A_m)$ is a rational form of either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$. More precisely,

THEOREM 3. *Let \mathfrak{h} be a 6-dimensional nilpotent Lie algebra of class 2 over \mathbf{Q} . Suppose that $[\mathfrak{h}, \mathfrak{h}]$ coincides with the 2-dimensional centre of \mathfrak{h} . Then $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$ for some $m \in \mathbf{Q}$ as above.*

Moreover,

- 1) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$ iff $m > 0$,
- 2) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$ iff $m = 0$,
- 3) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$ iff $m < 0$,

and up to isomorphism there are no more rational forms for \mathfrak{g}_- , \mathfrak{g}_0 , \mathfrak{g}_+ . The Lie algebras $\mathfrak{hei}_3(A_m)$ and $\mathfrak{hei}_3(A_n)$ are isomorphic over \mathbf{Q} if and only if A_m and A_n are isomorphic.

Proof. Take some \mathbf{Q} -basis x_1, \dots, x_6 of \mathfrak{h} . First of all, we may suppose that $[x_1, x_2] = x_5$ (possibly after a change of basis). Thus x_5 is central. We have to deal with two cases.

CASE 1. All brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) are multiples of x_5 . If $[x_1, x_j] = a_j x_5$, $[x_2, x_j] = b_j x_5$ then we set $X_j = x_j - a_j x_2 + b_j x_1$ whence $[x_1, X_j] = [x_2, X_j] = 0$.

Since $[\mathfrak{h}, \mathfrak{h}]$ is 2-dimensional we conclude that some commutator, say $[x_3, x_4]$, is not a multiple of x_5 (for convenience, we use lower-case ‘ x ’ instead of ‘ X ’). Consider

$$(4.1) \quad [x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

Commuting $[x_3, x_4]$ with x_1 , x_2 we obtain that $a = b = 0$. Let us suppose that $f = 0$. Then

$$(4.2) \quad [x_3, x_4] = cx_3 + dx_4 + ex_5.$$

Recall that x_5 and $[x_3, x_4]$ in the form (4.2) span the 2-dimensional centre. Commuting $cx_3 + dx_4 + ex_5$ from (4.2) with x_3 , x_4 we get $c = d = 0$ and a contradiction. Thus $f \neq 0$. We may assume that $[x_3, x_4] = x_6$ where x_6 is central. Hence, we have the following multiplication table for \mathfrak{h} : $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$, other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}).$$

CASE 2. Among the brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) there is at least one which is not a multiple of x_5 . In this case we may suppose (changing indices if necessary) that this bracket is $[x_1, x_3]$. Let

$$(4.3) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

and let us suppose that $d = f = 0$. Then

$$(4.4) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5.$$

Commuting the right-hand term of (4.4) with x_1 we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2x_3 + (ce + b)x_5.$$

Hence $c = b = 0$. By virtue of this $a = 0$ and we obtain a contradiction if we commute both sides of (4.4) with x_2 . It follows that either $d \neq 0$ or $f \neq 0$. In other words, we may suppose that $[x_1, x_3]$ is equal to x_6 .

Now

$$(4.5) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6$$

where x_5, x_6 span $[\mathfrak{h}, \mathfrak{h}]$. Suppose that $[x_2, x_3] = ax_5 + bx_6$. Adding if necessary some multiples of x_1 to x_2 and x_3 we obtain $[x_2, x_3] = 0$. In the same way we may suppose that $[x_1, x_4] = 0$. Adding to x_4 some multiple of x_1 we also obtain a relation $[x_2, x_4] = Cx_6$. Moreover, after scaling x_4 we get $C = 0$ or $C = 1$. Thus, \mathfrak{h} has a basis in which the *non-trivial* brackets are the following:

$$(4.6) \quad \begin{aligned} [x_1, x_2] &= x_5, & [x_1, x_3] &= x_6, \\ [x_2, x_4] &= Cx_6 \quad (C = 0 \text{ or } C = 1), & [x_3, x_4] &= Ax_5 + Bx_6. \end{aligned}$$

In any case $A^2 + B^2 + C^2 \neq 0$ because x_4 cannot belong to the 2-dimensional centre of \mathfrak{h} .

We will show that we can always make $C = 1$ and $B = 0$ in (4.6).

SUBCASE 2.1. If $C = 0$ then the following basis transformation

$$(4.7) \quad \begin{aligned} X_1 &= x_1, & X_2 &= ax_2 + x_3, \\ X_3 &= Ax_2 + Bx_3, & X_4 &= x_4, \end{aligned}$$

yields (a is any constant such that $aB \neq A$)

$$(4.8) \quad \begin{aligned} [X_1, X_2] &= ax_5 + x_6 = X_5, & [X_1, X_3] &= Ax_5 + Bx_6 = X_6, \\ [X_2, X_4] &= Ax_5 + Bx_6 = X_6, & [X_3, X_4] &= B(Ax_5 + Bx_6) = BX_6. \end{aligned}$$

From now on we may suppose that $C = 1$ in (4.6) and we arrive at

SUBCASE 2.2: $C = 1, A = 0$. Let

$$(4.9) \quad \begin{aligned} X_1 &= x_1 + ax_4, & X_2 &= x_2 - ax_3, \\ X_3 &= x_2 + dx_3, & X_4 &= -x_1 + dx_4, \end{aligned}$$

where $a, d, a + d \neq 0$, $aB \neq 1$, $dB \neq -1$. Hence

$$(4.10) \quad \begin{aligned} [X_1, X_2] &= x_5 + (a^2B - 2a)x_6 = X_5, \\ [X_1, X_3] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_2, X_4] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_3, X_4] &= x_5 + (d^2B + 2d)x_6 = \lambda X_5 + (1 - \lambda)X_6. \end{aligned}$$

Since a, d and $a + d$ are all non-zero, X_5 and X_6 are linearly independent. Straightforward computations yield

$$\lambda = \frac{dB + 1}{aB - 1} \neq 0, 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1: $C = 1$; $A, B, 4A + B^2 \neq 0$. Let now

$$(4.11) \quad \begin{aligned} X_1 &= x_1 + tx_4, & X_2 &= x_2 - tx_3, \\ X_3 &= x_3, & X_4 &= x_4, \end{aligned}$$

where $t = -B/2A$. Hence

$$(4.12) \quad \begin{aligned} [X_1, X_2] &= (1 + t^2A)x_5 + (t^2B - 2t)x_6 = X_5, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = \alpha X_5 = \frac{4A^2}{4A+B^2} X_5. \end{aligned}$$

SUBCASE 2.3.2: $C = 1$; $A, B \neq 0$, $4A + B^2 = 0$. The same transformation (4.11) yields

$$(4.13) \quad \begin{aligned} [X_1, X_2] &= 0, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = X_5 \end{aligned}$$

and, after the transformation $x_1 = X_3$, $x_2 = X_4$, $x_3 = X_1$, $x_4 = X_2$, $x_5 = X_5$, $x_6 = -X_6$, we obtain (4.12) with $\alpha = 0$. Anyway, we obtain the desired form of \mathfrak{h}

$$(4.14) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6, \quad [x_2, x_4] = x_6, \quad [x_3, x_4] = Ax_5.$$

Scaling x_3 , x_4 by $\lambda \neq 0$ we may suppose that $A = m$ where m is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism $\rho: \mathfrak{h} \rightarrow \mathfrak{hei}_3(A_m)$. Recall that A_m has a basis $1, x$ over \mathbf{Q} such that $x^2 = m$. Here are the matrices representing $\rho(x_i)$ if $m \neq 1$ (the case $m = 1$ is left to the reader as an easy exercise):

$$(4.15) \quad \begin{aligned} \rho(x_1) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_5) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \rho(x_3) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_4) &= \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_6) &= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now it is evident that $\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to either $\mathfrak{hei}_3(\mathbb{R}) \oplus \mathfrak{hei}_3(\mathbb{R})$, or $\mathfrak{hei}_3(\mathbb{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbb{C})$ depending on the sign of m . Thus, we have classified up to \mathbb{Q} -isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete. \square

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over \mathbb{R} . Indeed, the centralizer of any element in $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbb{C})$ is even dimensional over \mathbb{R} since this algebra can be viewed as a complex Lie algebra, whereas in both $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbb{R}) \oplus \mathfrak{hei}_3(\mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbb{R}[x]/(x^2))$ there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer $C(x)$ will not be changed if we scale x by any $\lambda \neq 0$ or add to x any central element. This means that dimension of the centralizer is a well-defined function on the projective space $\mathbf{P}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ where \mathfrak{g} is either \mathfrak{g}_+ or \mathfrak{g}_0 . Straightforward computations show that in $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0])$ all points with 5-dimensional centralizer belong to a unique line whereas in $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+])$ the points under consideration form two disjoint lines.

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