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#### 4. NILPOTENT LIE ALGEBRAS WITH INFINITELY MANY NON-ISOMORPHIC RATIONAL FORMS

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

##### 4.1 BASIC LEMMA

Let

$$\mathfrak{h} = \bigoplus_{i=1}^c \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over  $\mathbf{Q}$  generated by  $\mathfrak{h}_1$ . Let  $\mathbf{K}$  be a number field,  $\dim_{\mathbf{Q}} \mathbf{K} = d$ , of type  $(s, t)$ , that is, there are  $s$  real and  $2t$  complex embeddings of  $\mathbf{K}$  in  $\mathbf{C}$  ( $d = s + 2t$ ) whence there exists an isomorphism of  $\mathbf{R}$ -algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^s \mathbf{R} \oplus \bigoplus_{l=1}^t \mathbf{C}.$$

More generally one can take a finite-dimensional commutative associative algebra  $\mathbf{A}$  over  $\mathbf{Q}$  instead of  $\mathbf{K}$ . We consider the Lie algebra  $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$  as a Lie algebra over  $\mathbf{Q}$ . This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^s \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^t \mathfrak{h}(\mathbf{C}),$$

i.e.,  $\mathfrak{h}(\mathbf{K})$  is a  $\mathbf{Q}$ -form of the last Lie algebra for any number field  $\mathbf{K}$  of type  $(s, t)$ . Secondly, there is an embedding  $R: \mathbf{K}^* \rightarrow \text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$  of the multiplicative group  $\mathbf{K}^*$  such that  $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$  where  $h_i \in \mathfrak{h}_i$  is homogenous of degree  $i$ . The following lemma is straightforward.

**LEMMA 4.1.** *Let  $\mathbf{K} \neq \mathbf{K}'$  be two distinct number fields of the same type. If there is no injection of  $\mathbf{K}^*$  into  $\text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$  then two  $\mathbf{Q}$ -forms  $\mathfrak{h}(\mathbf{K})$  and  $\mathfrak{h}(\mathbf{K}')$  are not isomorphic.*

##### 4.2 PROOF OF THEOREM 2

We start with the class of nilpotence  $c = 2$ . Let  $\mathbf{K} = \mathbf{Q}(\sqrt{m})$  and  $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$ , where  $m \neq n$  are two positive (resp. negative) square-free integers. Consider the automorphism  $A = R(\sqrt{m})$  of  $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p, \mathbf{K})$ . One immediately checks that

- 1)  $A^2$  acts on  $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]$  as  $m \cdot \text{Id}$ ;
- 2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]} = m \cdot \text{Id}.$$

By Lemma 4.1 we must prove that there is no such automorphism for  $\mathfrak{h}(\mathbf{K}') = \mathfrak{h}(\mathbf{Q}(\sqrt{n}))$ . We choose the following basis of  $\mathfrak{h}(\mathbf{K}')$  over  $\mathbf{Q}$ :

$$X_i = x_i \otimes 1, \quad Y_i = x_i \otimes \sqrt{n}, \quad C_{ij} = c_{ij} \otimes 1, \quad Z_{ij} = c_{ij} \otimes \sqrt{n},$$

$x_1, \dots, x_p, c_{ij} = [x_i, x_j]$  being the standard basis of  $\mathfrak{f}_2(p, \mathbf{Q})$ .

Suppose that there exists an automorphism  $A'$  with two above properties. First of all, let us show that  $[X_i, A'(X_i)] = 0$ . On the one hand,

$$A'[X_i, A'(X_i)] = [A'(X_i), mX_i] = -m[X_i, A'(X_i)].$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)].$$

Since the centralizer of  $X_i$  is generated modulo the centre by  $X_i, Y_i$  it follows that

$$A'(X_i) = p_i X_i + q_i Y_i + \varepsilon = x_i \otimes (p_i + q_i \sqrt{n}) + \varepsilon, \quad q_i \neq 0.$$

Here  $\varepsilon$  stands for a central element which plays no role below.

Consider now  $[X_i, A'(X_j)] = c_{ij} \otimes (p_j + q_j \sqrt{n})$ . On the one hand,

$$A'[X_i, A'(X_j)] = [A'(X_i), mX_j] = c_{ij} \otimes m(p_i + q_i \sqrt{n}).$$

On the other hand,

$$A'[X_i, A'(X_j)] = m[X_i, A'(X_j)] = c_{ij} \otimes m(p_j + q_j \sqrt{n}),$$

whence

$$p_i + q_i \sqrt{n} = p_j + q_j \sqrt{n} = p + q \sqrt{n} \notin \mathbf{Q} \quad \forall i, j.$$

Finally, we apply  $A'$  to  $[A'(X_i), A'(X_j)] = c_{ij} \otimes (p + q \sqrt{n})^2$ . On the one hand,

$$A'[A'(X_i), A'(X_j)] = [mX_i, mX_j] = c_{ij} \otimes m^2.$$

On the other hand,

$$A'[A'(X_i), A'(X_j)] = m[A'(X_i), A'(X_j)] = c_{ij} \otimes m(p + q \sqrt{n})^2.$$

It follows that  $m = (p + q \sqrt{n})^2$ . We have obtained a contradiction since  $q \neq 0$ . Thus, there are infinitely many non-isomorphic rational forms of  $\mathfrak{f}_2(p, \mathbf{R}) \oplus \mathfrak{f}_2(p, \mathbf{R})$  and of  $\mathfrak{f}_2(p, \mathbf{C})$ .

More generally let  $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$  be a free nilpotent Lie algebra of class  $c \geq 3$  on  $p$  generators. Then  $\mathfrak{g} \oplus \mathfrak{g}$  and  $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$  (as a Lie algebra over  $\mathbf{R}$ ) also have infinitely many non-isomorphic rational forms. Consider the automorphism  $A$  as above and note that it respects the descending central series. Any isomorphism between  $\mathfrak{f}_c(p, \mathbf{K})$  and  $\mathfrak{f}_c(p, \mathbf{K}')$  must respect it, too. Then we can take the free nilpotent quotients of class 2 of both algebras and obtain a contradiction just like in the first part of the proof.  $\square$

Thus, the case of a free nilpotent Lie algebra  $\mathfrak{f}_c(p, \mathbf{C})$  (as a Lie algebra over  $\mathbf{R}$ ) on  $p$  generators differs from the case 2.2.

REMARK. All rational forms of  $\mathfrak{f}_2(2, \mathbf{C}) = \text{hei}_3(\mathbf{C})$  and  $\mathfrak{f}_2(2, \mathbf{R}) \oplus \mathfrak{f}_2(2, \mathbf{R}) = \text{hei}_3(\mathbf{R}) \oplus \text{hei}_3(\mathbf{R})$  are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type  $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$  where  $F_c(p, \mathbf{R})$  is the free nilpotent Lie group on  $p$  free generators.*

#### 4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let  $m$  be a rational number and  $A_m = \mathbf{Q}[x]/(x^2 - m)$ .  $A_m$  is a 2-dimensional commutative algebra over  $\mathbf{Q}$  which depends only on  $m$  modulo square factors. Thus there are four types of  $A_m$ :

- 1) if  $m = 1$  then  $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$ ;
- 2) if  $m > 1$  is a positive square-free integer then  $A_m \cong \mathbf{Q}(\sqrt{m})$  is a real quadratic field over  $\mathbf{Q}$ ;
- 3) if  $m = 0$  then  $A_0$  is the algebra of dual numbers over  $\mathbf{Q}$ ;
- 4) if  $m$  is a negative square-free integer then  $A_m \cong \mathbf{Q}(\sqrt{m})$  is an imaginary quadratic field over  $\mathbf{Q}$ .

Let  $\text{hei}_3(A_m)$  be a Heisenberg algebra over  $A_m$  considered over  $\mathbf{Q}$ . Then  $\text{hei}_3(A_m)$  is a rational form of either  $\text{hei}_3(\mathbf{R}) \oplus \text{hei}_3(\mathbf{R})$ , or  $\text{hei}_3(\mathbf{R}[x]/(x^2))$ , or  $\text{hei}_3(\mathbf{C})$ . More precisely,

**THEOREM 3.** *Let  $\mathfrak{h}$  be a 6-dimensional nilpotent Lie algebra of class 2 over  $\mathbf{Q}$ . Suppose that  $[\mathfrak{h}, \mathfrak{h}]$  coincides with the 2-dimensional centre of  $\mathfrak{h}$ . Then  $\mathfrak{h} \cong \text{hei}_3(A_m)$  for some  $m \in \mathbf{Q}$  as above.*

Moreover,

- 1)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \text{hei}_3(\mathbf{R}) \oplus \text{hei}_3(\mathbf{R}) = \mathfrak{g}_+$  iff  $m > 0$ ,
- 2)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \text{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$  iff  $m = 0$ ,
- 3)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \text{hei}_3(\mathbf{C}) = \mathfrak{g}_-$  iff  $m < 0$ ,

and up to isomorphism there are no more rational forms for  $\mathfrak{g}_-$ ,  $\mathfrak{g}_0$ ,  $\mathfrak{g}_+$ . The Lie algebras  $\text{hei}_3(A_m)$  and  $\text{hei}_3(A_n)$  are isomorphic over  $\mathbf{Q}$  if and only if  $A_m$  and  $A_n$  are isomorphic.