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4. NILPOTENT LIE ALGEBRAS WITH INFINITELY MANY NON-ISOMORPHIC RATIONAL FORMS

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 BASIC LEMMA

Let

$$\mathfrak{h} = \bigoplus_{i=1}^c \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over \mathbf{Q} generated by \mathfrak{h}_1 . Let \mathbf{K} be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s, t) , that is, there are s real and $2t$ complex embeddings of \mathbf{K} in \mathbf{C} ($d = s + 2t$) whence there exists an isomorphism of \mathbf{R} -algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^s \mathbf{R} \oplus \bigoplus_{l=1}^t \mathbf{C}.$$

More generally one can take a finite-dimensional commutative associative algebra \mathbf{A} over \mathbf{Q} instead of \mathbf{K} . We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over \mathbf{Q} . This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^s \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^t \mathfrak{h}(\mathbf{C}),$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a \mathbf{Q} -form of the last Lie algebra for any number field \mathbf{K} of type (s, t) . Secondly, there is an embedding $R: \mathbf{K}^* \rightarrow \text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group \mathbf{K}^* such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenous of degree i . The following lemma is straightforward.

LEMMA 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of \mathbf{K}^* into $\text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two \mathbf{Q} -forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 PROOF OF THEOREM 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p, \mathbf{K})$. One immediately checks that

- 1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]$ as $m \cdot \text{Id}$;
- 2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]} = m \cdot \text{Id}.$$