

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS
Autor: Semenov, Yu. S.
Kapitel: 4.1 Basic lemma
DOI: <https://doi.org/10.5169/seals-66073>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4. NILPOTENT LIE ALGEBRAS WITH INFINITELY MANY
NON-ISOMORPHIC RATIONAL FORMS

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 BASIC LEMMA

Let

$$\mathfrak{h} = \bigoplus_{i=1}^c \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over \mathbf{Q} generated by \mathfrak{h}_1 . Let \mathbf{K} be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s, t) , that is, there are s real and $2t$ complex embeddings of \mathbf{K} in \mathbf{C} ($d = s + 2t$) whence there exists an isomorphism of \mathbf{R} -algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^s \mathbf{R} \oplus \bigoplus_{l=1}^t \mathbf{C}.$$

More generally one can take a finite-dimensional commutative associative algebra \mathbf{A} over \mathbf{Q} instead of \mathbf{K} . We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over \mathbf{Q} . This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^s \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^t \mathfrak{h}(\mathbf{C}),$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a \mathbf{Q} -form of the last Lie algebra for any number field \mathbf{K} of type (s, t) . Secondly, there is an embedding $R: \mathbf{K}^* \rightarrow \text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group \mathbf{K}^* such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenous of degree i . The following lemma is straightforward.

LEMMA 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of \mathbf{K}^* into $\text{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two \mathbf{Q} -forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 PROOF OF THEOREM 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p, \mathbf{K})$. One immediately checks that

- 1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]$ as $m \cdot Id$;
- 2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]} = m \cdot Id.$$