4. Nilpotent Lie algebras with infinitely many non-isomorphic rational forms

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 Basic lemma

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{c} \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over $\mathbf{Q}$ generated by $\mathfrak{h}_1$. Let $\mathbf{K}$ be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s,t), that is, there are s real and $2t$ complex embeddings of $\mathbf{K}$ in $\mathbf{C}$ $(d = s + 2t)$ whence there exists an isomorphism of $\mathbf{R}$-algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^{s} \mathbf{R} \oplus \bigoplus_{l=1}^{t} \mathbf{C}\,.$$

More generally one can take a finite-dimensional commutative associative algebra $\mathbf{A}$ over $\mathbf{Q}$ instead of $\mathbf{K}$. We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over $\mathbf{Q}$. This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^{s} \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^{t} \mathfrak{h}(\mathbf{C})\,,$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a $\mathbf{Q}$-form of the last Lie algebra for any number field $\mathbf{K}$ of type (s,t). Secondly, there is an embedding $R\colon \mathbf{K}^* \to \mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group $\mathbf{K}^*$ such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenenous of degree i. The following lemma is straightforward.

Lemma 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of $\mathbf{K}^*$ into $\mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two $\mathbf{Q}$-forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 Proof of Theorem 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p, \mathbf{K})$. One immediately checks that

1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]$ as $m \cdot Id$;

2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]} = m \cdot Id\,.$$