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# 4. NILPOTENT LIE ALGEBRAS WITH INFINITELY MANY NON-ISOMORPHIC RATIONAL FORMS

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

# 4.1 Basic Lemma

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{c} \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over  $\mathbf{Q}$  generated by  $\mathfrak{h}_1$ . Let  $\mathbf{K}$  be a number field,  $\dim_{\mathbf{Q}} \mathbf{K} = d$ , of type (s,t), that is, there are s real and 2t complex embeddings of  $\mathbf{K}$  in  $\mathbf{C}$  (d=s+2t) whence there exists an isomorphism of  $\mathbf{R}$ -algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^{s} \mathbf{R} \oplus \bigoplus_{l=1}^{t} \mathbf{C}$$
.

More generally one can take a finite-dimensional commutative associative algebra A over Q instead of K. We consider the Lie algebra  $\mathfrak{h}(K) = \mathfrak{h} \otimes_Q K$  as a Lie algebra over Q. This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^{s} \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^{t} \mathfrak{h}(\mathbf{C}),$$

i.e.,  $\mathfrak{h}(\mathbf{K})$  is a **Q**-form of the last Lie algebra for any number field **K** of type (s,t). Secondly, there is an embedding  $R \colon \mathbf{K}^* \to \operatorname{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$  of the multiplicative group  $\mathbf{K}^*$  such that  $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$  where  $h_i \in \mathfrak{h}_i$  is homogeneous of degree i. The following lemma is straightforward.

LEMMA 4.1. Let  $\mathbf{K} \neq \mathbf{K}'$  be two distinct number fields of the same type. If there is no injection of  $\mathbf{K}^*$  into  $\operatorname{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$  then two  $\mathbf{Q}$ -forms  $\mathfrak{h}(\mathbf{K})$  and  $\mathfrak{h}(\mathbf{K}')$  are not isomorphic.

# 4.2 Proof of Theorem 2

We start with the class of nilpotence c=2. Let  $\mathbf{K}=\mathbf{Q}(\sqrt{m})$  and  $\mathbf{K}'=\mathbf{Q}(\sqrt{n})$ , where  $m\neq n$  are two positive (resp. negative) square-free integers. Consider the automorphism  $A=R(\sqrt{m})$  of  $\mathfrak{h}(\mathbf{K})=\mathfrak{f}_2(p,\mathbf{K})$ . One immediately checks that

- 1)  $A^2$  acts on  $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]$  as  $m \cdot Id$ ;
- 2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]} = m \cdot Id$$
.

By Lemma 4.1 we must prove that there is no such automorphism for  $\mathfrak{h}(\mathbf{K}') = \mathfrak{h}(\mathbf{Q}(\sqrt{n}))$ . We choose the following basis of  $\mathfrak{h}(\mathbf{K}')$  over  $\mathbf{Q}$ :

$$X_i = x_i \otimes 1$$
,  $Y_i = x_i \otimes \sqrt{n}$ ,  $C_{ij} = c_{ij} \otimes 1$ ,  $Z_{ij} = c_{ij} \otimes \sqrt{n}$ ,

 $x_1, \ldots, x_p, c_{ij} = [x_i, x_j]$  being the standard basis of  $\mathfrak{f}_2(p, \mathbf{Q})$ .

Suppose that there exists an automorphism A' with two above properties. First of all, let us show that  $[X_i, A'(X_i)] = 0$ . On the one hand,

$$A'[X_i, A'(X_i)] = [A'(X_i), mX_i] = -m[X_i, A'(X_i)].$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)].$$

Since the centralizer of  $X_i$  is generated modulo the centre by  $X_i$ ,  $Y_i$  it follows that

$$A'(X_i) = p_i X_i + q_i Y_i + \varepsilon = x_i \otimes (p_i + q_i \sqrt{n}) + \varepsilon, \quad q_i \neq 0.$$

Here  $\varepsilon$  stands for a central element which plays no role below.

Consider now  $[X_i, A'(X_j)] = c_{ij} \otimes (p_j + q_j \sqrt{n})$ . On the one hand,

$$A'[X_i, A'(X_j)] = [A'(X_i), mX_j] = c_{ij} \otimes m(p_i + q_i \sqrt{n}).$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)] = c_{ii} \otimes m(p_i + q_i \sqrt{n}),$$

whence

$$p_i + q_i \sqrt{n} = p_j + q_i \sqrt{n} = p + q \sqrt{n} \notin \mathbf{Q} \ \forall i, j.$$

Finally, we apply A' to  $[A'(X_i), A'(X_j)] = c_{ij} \otimes (p + q\sqrt{n})^2$ . On the one hand,

$$A'[A'(X_i),A'(X_j)]=[mX_i,mX_j]=c_{ij}\otimes m^2.$$

On the other hand,

$$A'[A'(X_i), A'(X_i)] = m[A'(X_i), A'(X_j)] = c_{ij} \otimes m(p + q\sqrt{n})^2.$$

It follows that  $m=(p+q\sqrt{n})^2$ . We have obtained a contradiction since  $q\neq 0$ . Thus, there are infinitely many non-isomorphic rational forms of  $\mathfrak{f}_2(p,\mathbf{R})\oplus\mathfrak{f}_2(p,\mathbf{R})$  and of  $\mathfrak{f}_2(p,\mathbf{C})$ .

More generally let  $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$  be a free nilpotent Lie algebra of class  $c \geq 3$  on p generators. Then  $\mathfrak{g} \oplus \mathfrak{g}$  and  $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$  (as a Lie algebra over  $\mathbf{R}$ ) also have infinitely many non-isomorphic rational forms. Consider the automorphism A as above and note that it respects the descending central series. Any isomorphism between  $\mathfrak{f}_c(p, \mathbf{K})$  and  $\mathfrak{f}_c(p, \mathbf{K}')$  must respect it, too. Then we can take the free nilpotent quotients of class 2 of both algebras and obtain a contradiction just like in the first part of the proof.  $\square$ 

Thus, the case of a free nilpotent Lie algebra  $f_c(p, \mathbb{C})$  (as a Lie algebra over  $\mathbb{R}$ ) on p generators differs from the case 2.2.

REMARK. All rational forms of  $\mathfrak{f}_2(2, \mathbb{C}) = \mathfrak{hei}_3(\mathbb{C})$  and  $\mathfrak{f}_2(2, \mathbb{R}) \oplus \mathfrak{f}_2(2, \mathbb{R}) = \mathfrak{hei}_3(\mathbb{R}) \oplus \mathfrak{hei}_3(\mathbb{R})$  are listed in Theorem 3.

COROLLARY 4.2. There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type  $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$  where  $F_c(p, \mathbf{R})$  is the free nilpotent Lie group on p free generators.

4.3 Classification of rational forms for some 6-dimensional Lie algebras

Let m be a rational number and  $A_m = \mathbf{Q}[x]/(x^2 - m)$ .  $A_m$  is a 2-dimensional commutative algebra over  $\mathbf{Q}$  which depends only on m modulo square factors. Thus there are four types of  $A_m$ :

- 1) if m = 1 then  $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$ ;
- 2) if m > 1 is a positive square-free integer then  $A_m \cong \mathbb{Q}(\sqrt{m})$  is a real quadratic field over  $\mathbb{Q}$ ;
- 3) if m = 0 then  $A_0$  is the algebra of dual numbers over  $\mathbf{Q}$ ;
- 4) if m is a negative square-free integer then  $A_m \cong \mathbb{Q}(\sqrt{m})$  is an imaginary quadratic field over  $\mathbb{Q}$ .

Let  $\mathfrak{hei}_3(A_m)$  be a Heisenberg algebra over  $A_m$  considered over  $\mathbb{Q}$ . Then  $\mathfrak{hei}_3(A_m)$  is a rational form of either  $\mathfrak{hei}_3(\mathbb{R}) \oplus \mathfrak{hei}_3(\mathbb{R})$ , or  $\mathfrak{hei}_3(\mathbb{R}[x]/(x^2))$ , or  $\mathfrak{hei}_3(\mathbb{C})$ . More precisely,

THEOREM 3. Let  $\mathfrak{h}$  be a 6-dimensional nilpotent Lie algebra of class 2 over  $\mathbb{Q}$ . Suppose that  $[\mathfrak{h},\mathfrak{h}]$  coincides with the 2-dimensional centre of  $\mathfrak{h}$ . Then  $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$  for some  $m \in \mathbb{Q}$  as above.

Moreover,

- $1) \quad \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+ \quad \textit{iff} \quad m > 0 \,,$
- 2)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$  iff m = 0,
- 3)  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$  iff m < 0,

and up to isomorphism there are no more rational forms for  $\mathfrak{g}_-$ ,  $\mathfrak{g}_0$ ,  $\mathfrak{g}_+$ . The Lie algebras  $\mathfrak{hei}_3(A_m)$  and  $\mathfrak{hei}_3(A_n)$  are isomorphic over  $\mathbf{Q}$  if and only if  $A_m$  and  $A_n$  are isomorphic.

*Proof.* Take some **Q**-basis  $x_1, \ldots, x_6$  of  $\mathfrak{h}$ . First of all, we may suppose that  $[x_1, x_2] = x_5$  (possibly after a change of basis). Thus  $x_5$  is central. We have to deal with two cases.

CASE 1. All brackets  $[x_1, x_j]$ ,  $[x_2, x_j]$   $(j \ge 3)$  are multiples of  $x_5$ . If  $[x_1, x_j] = a_j x_5$ ,  $[x_2, x_j] = b_j x_5$  then we set  $X_j = x_j - a_j x_2 + b_j x_1$  whence  $[x_1, X_j] = [x_2, X_j] = 0$ .

Since  $[\mathfrak{h},\mathfrak{h}]$  is 2-dimensional we conclude that some commutator, say  $[x_3,x_4]$ , is not a multiple of  $x_5$  (for convenience, we use lower-case 'x' instead of 'X'). Consider

$$[x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

Commuting  $[x_3, x_4]$  with  $x_1$ ,  $x_2$  we obtain that a = b = 0. Let us suppose that f = 0. Then

$$[x_3, x_4] = cx_3 + dx_4 + ex_5.$$

Recall that  $x_5$  and  $[x_3, x_4]$  in the form (4.2) span the 2-dimensional centre. Commuting  $cx_3 + dx_4 + ex_5$  from (4.2) with  $x_3$ ,  $x_4$  we get c = d = 0 and a contradiction. Thus  $f \neq 0$ . We may assume that  $[x_3, x_4] = x_6$  where  $x_6$  is central. Hence, we have the following multiplication table for  $\mathfrak{h}: [x_1, x_2] = x_5$ ,  $[x_3, x_4] = x_6$ , other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}).$$

CASE 2. Among the brackets  $[x_1, x_j]$ ,  $[x_2, x_j]$   $(j \ge 3)$  there is at least one which is not a multiple of  $x_5$ . In this case we may suppose (changing indices if necessary) that this bracket is  $[x_1, x_3]$ . Let

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6$$

and let us suppose that d = f = 0. Then

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5.$$

Commuting the right-hand term of (4.4) with  $x_1$  we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2x_3 + (ce + b)x_5.$$

Hence c = b = 0. By virtue of this a = 0 and we obtain a contradiction if we commute both sides of (4.4) with  $x_2$ . It follows that either  $d \neq 0$  or  $f \neq 0$ . In other words, we may suppose that  $[x_1, x_3]$  is equal to  $x_6$ .

Now

$$[x_1, x_2] = x_5, \ [x_1, x_3] = x_6$$

where  $x_5$ ,  $x_6$  span  $[\mathfrak{h},\mathfrak{h}]$ . Suppose that  $[x_2,x_3]=ax_5+bx_6$ . Adding if necessary some multiples of  $x_1$  to  $x_2$  and  $x_3$  we obtain  $[x_2,x_3]=0$ . In the same way we may suppose that  $[x_1,x_4]=0$ . Adding to  $x_4$  some multiple of  $x_1$  we also obtain a relation  $[x_2,x_4]=Cx_6$ . Moreover, after scaling  $x_4$  we get C=0 or C=1. Thus,  $\mathfrak{h}$  has a basis in which the *non-trivial* brackets are the following:

(4.6) 
$$[x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = Cx_6 (C = 0 mtext{ or } C = 1), [x_3, x_4] = Ax_5 + Bx_6.$$

In any case  $A^2 + B^2 + C^2 \neq 0$  because  $x_4$  cannot belong to the 2-dimensional centre of  $\mathfrak{h}$ .

We will show that we can always make C = 1 and B = 0 in (4.6).

SUBCASE 2.1. If C = 0 then the following basis transformation

(4.7) 
$$X_1 = x_1, X_2 = ax_2 + x_3, X_3 = Ax_2 + Bx_3, X_4 = x_4,$$

yields (a is any constant such that  $aB \neq A$ )

(4.8) 
$$[X_1, X_2] = ax_5 + x_6 = X_5, \qquad [X_1, X_3] = Ax_5 + Bx_6 = X_6, [X_2, X_4] = Ax_5 + Bx_6 = X_6, \qquad [X_3, X_4] = B(Ax_5 + Bx_6) = BX_6.$$

From now on we may suppose that C = 1 in (4.6) and we arrive at

SUBCASE 2.2: 
$$C = 1$$
,  $A = 0$ . Let  $X_1 = x_1 + ax_4$ ,  $X_2 = x_2 - ax_3$ ,  $X_3 = x_2 + dx_3$ ,  $X_4 = -x_1 + dx_4$ ,

where  $a, d, a + d \neq 0$ ,  $aB \neq 1$ ,  $dB \neq -1$ . Hence

$$[X_{1}, X_{2}] = x_{5} + (a^{2}B - 2a)x_{6} = X_{5},$$

$$[X_{1}, X_{3}] = x_{5} + (d - a - adB)x_{6} = X_{6},$$

$$[X_{2}, X_{4}] = x_{5} + (d - a - adB)x_{6} = X_{6},$$

$$[X_{3}, X_{4}] = x_{5} + (d^{2}B + 2d)x_{6} = \lambda X_{5} + (1 - \lambda)X_{6}.$$

Since a, d and a+d are all non-zero,  $X_5$  and  $X_6$  are linearly independent. Straightforward computations yield

$$\lambda = \frac{dB+1}{aB-1} \neq 0, 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1: C = 1;  $A, B, 4A + B^2 \neq 0$ . Let now

(4.11) 
$$X_1 = x_1 + tx_4, \qquad X_2 = x_2 - tx_3, \\ X_3 = x_3, \qquad X_4 = x_4,$$

where t = -B/2A. Hence

$$[X_{1}, X_{2}] = (1 + t^{2}A)x_{5} + (t^{2}B - 2t)x_{6} = X_{5},$$

$$[X_{1}, X_{3}] = -tAx_{5} + (1 - tB)x_{6} = X_{6},$$

$$[X_{2}, X_{4}] = -tAx_{5} + (1 - tB)x_{6} = X_{6},$$

$$[X_{3}, X_{4}] = Ax_{5} + Bx_{6} = \alpha X_{5} = \frac{4A^{2}}{4A + B^{2}}X_{5}.$$

SUBCASE 2.3.2: C=1;  $A,B\neq 0$ ,  $4A+B^2=0$ . The same transformation (4.11) yields

$$[X_{1}, X_{2}] = 0,$$

$$[X_{1}, X_{3}] = -tAx_{5} + (1 - tB)x_{6} = X_{6},$$

$$[X_{2}, X_{4}] = -tAx_{5} + (1 - tB)x_{6} = X_{6},$$

$$[X_{3}, X_{4}] = Ax_{5} + Bx_{6} = X_{5}$$

and, after the transformation  $x_1 = X_3$ ,  $x_2 = X_4$ ,  $x_3 = X_1$ ,  $x_4 = X_2$ ,  $x_5 = X_5$ ,  $x_6 = -X_6$ , we obtain (4.12) with  $\alpha = 0$ . Anyway, we obtain the desired form of  $\mathfrak{h}$ 

$$(4.14) [x1, x2] = x5, [x1, x3] = x6, [x2, x4] = x6, [x3, x4] = Ax5.$$

Scaling  $x_3$ ,  $x_4$  by  $\lambda \neq 0$  we may suppose that A = m where m is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism  $\rho: \mathfrak{h} \to \mathfrak{hei}_3(A_m)$ . Recall that  $A_m$  has a basis 1, x over  $\mathbb{Q}$  such that  $x^2 = m$ . Here are the matrices representing  $\rho(x_i)$  if  $m \neq 1$  (the case m = 1 is left to the reader as an easy exercise):

$$\rho(x_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(4.15)$$

$$\rho(x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_4) = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_6) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now it is evident that  $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R}$  is isomorphic to either  $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ , or  $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ , or  $\mathfrak{hei}_3(\mathbf{C})$  depending on the sign of m. Thus, we have classified up to  $\mathbf{Q}$ -isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete.  $\square$ 

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over  $\mathbf{R}$ . Indeed, the centralizer of any element in  $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbf{C})$  is even dimensional over  $\mathbf{R}$  since this algebra can be viewed as a complex Lie algebra, whereas in both  $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$  and  $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$  there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer C(x) will not be changed if we scale x by any  $\lambda \neq 0$  or add to x any central element. This means that dimension of the centralizer is a well-defined function on the projective space  $\mathbf{P}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$  where  $\mathfrak{g}$  is either  $\mathfrak{g}_+$  or  $\mathfrak{g}_0$ . Straightforward computations show that in  $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0])$  all points with 5-dimensional centralizer belong to a unique line whereas in  $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+,\mathfrak{g}_+])$  the points under consideration form two disjoint lines.

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