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3. Real Lie algebras  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  are not isomorphic but  $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$ .
4. If  $t \in \mathbf{Q} \setminus \{0\}$  then the rational algebra  $\mathfrak{g}_t \cong \mathfrak{g}_1$  over  $\mathbf{Q}$ .
5.  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  are two Lie algebras with a unique rational form up to isomorphism.
6. Let  $\mathfrak{g}$  be a split real simple Lie algebra of type  $G_2$ ,  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  be the triangular decomposition of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Then  $\mathfrak{n}_+$  is isomorphic to  $\mathfrak{g}_0$ .

### 3. MALCEV'S EXAMPLE

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a  $\mathbf{Q}$ -isomorphism between  $\mathfrak{g}_t$  and  $\mathfrak{g}_s$ . It must be written in the following form (cf. [5]) since  $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$ ,  $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$  and the centralizer  $\mathfrak{c}$  of  $C^2\mathfrak{g}$ , which is an ideal in this case, is spanned by  $x_3, \dots, x_6$ .

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = \phantom{a_{21}x_1 + a_{22}x_2} + a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = \phantom{a_{21}x_1 + a_{22}x_2} + \phantom{a_{33}x_3} + a_{44}x_4 + \dots \end{cases}$$

We do not explicit the expressions for  $y_5, y_6$ . Here  $y_1, \dots, y_6$  are basis elements of  $\mathfrak{g}_s$  satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$ . On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$(3.1) \quad \begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} (3.2) \quad [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned}$$

Let  $u = a_{12}a_{33} + a_{11}a_{34}$ ,  $v = a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33}$ . In view of (3.1) and (3.2) we have

$$(a_{22}y_5 - a_{12}y_6)u/\Delta^2 + (a_{11}y_6 - a_{21}y_5)v/\Delta^2 = y_6,$$

whence

$$(3.3) \quad \begin{cases} va_{11} - ua_{12} = \Delta^2, \\ va_{21} - ua_{22} = 0. \end{cases}$$

It follows that

$$(3.4) \quad \begin{cases} u = a_{21}\Delta, \\ v = a_{22}\Delta. \end{cases}$$

In addition,

$$(3.5) \quad \begin{aligned} [y_2, y_3] &= a_{21}a_{33}[x_1, x_3] + a_{22}a_{33}[x_2, x_3] + a_{21}a_{34}[x_1, x_4] + a_{22}a_{34}[x_2, x_4] \\ &= a_{21}a_{33}x_6 + a_{22}a_{33}(x_5 + tx_6) + a_{21}a_{34}x_5 + a_{22}a_{34}x_6 \\ &= (a_{22}a_{33} + a_{21}a_{34})x_5 + (a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33})x_6 = y_5 + sy_6. \end{aligned}$$

Let  $p = a_{22}a_{33} + a_{21}a_{34}$ ,  $q = a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33}$ . In view of (3.1), (3.5)

$$(a_{22}y_5 - a_{12}y_6)p/\Delta^2 + (a_{11}y_6 - a_{21}y_5)q/\Delta^2 = y_5 + sy_6.$$

This implies that

$$(3.6) \quad \begin{cases} qa_{11} - pa_{12} = s\Delta^2, \\ qa_{21} - pa_{22} = -\Delta^2. \end{cases}$$

Consequently,

$$(3.7) \quad \begin{cases} p = (sa_{21} + a_{11})\Delta, \\ q = (sa_{22} + a_{12})\Delta. \end{cases}$$

Substituting  $u, v, p, q$  by the expressions given in (3.4), (3.7) we conclude that

$$(3.8) \quad \begin{cases} a_{11}a_{34} + a_{12}a_{33} = a_{21}\Delta, \\ a_{11}a_{33} + a_{12}(a_{34} + ta_{33}) = a_{22}\Delta, \\ a_{21}a_{34} + a_{22}a_{33} = (a_{11} + sa_{21})\Delta, \\ a_{21}a_{33} + a_{22}(a_{34} + ta_{33}) = (a_{12} + sa_{22})\Delta. \end{cases}$$

The first and the third equations of (3.8) yield

$$(3.9) \quad \begin{cases} a_{34} = a_{21}a_{22} - a_{12}(a_{11} + sa_{21}), \\ a_{33} = a_{11}(a_{11} + sa_{21}) - a_{21}a_{22}. \end{cases}$$

The two remaining ones yield

$$(3.10) \quad \begin{cases} a_{33} = a_{22}a_{22} - a_{12}(a_{12} + sa_{22}), \\ a_{34} + ta_{33} = a_{11}(a_{12} + sa_{22}) - a_{21}a_{21}, \end{cases}$$

whence

$$(3.11) \quad \begin{cases} a_{11}^2 + sa_{11}a_{21} - a_{21}^2 = a_{22}^2 - sa_{22}a_{12} - a_{12}^2 \neq 0, \\ 2(a_{11}a_{12} - a_{21}a_{22}) + s(a_{11}a_{22} - a_{21}a_{12}) = t(a_{11}^2 + sa_{11}a_{21} - a_{21}^2). \end{cases}$$

Let

$$\begin{cases} x_{11} = a_{11} + sa_{21}/2, \\ x_{12} = a_{12} + sa_{22}/2, \\ x_{21} = a_{21}, \\ x_{22} = a_{22}. \end{cases}$$

The system (3.11) can be rewritten in the form

$$(3.12) \quad \begin{cases} x_{11}^2 - (1 + s^2/4)x_{21}^2 = -(x_{12}^2 - (1 + s^2/4)x_{22}^2) \neq 0, \\ 2(x_{11}x_{12} - (1 + s^2/4)x_{21}x_{22}) = t(x_{11}^2 - (1 + s^2/4)x_{21}^2). \end{cases}$$

Thus we may conclude that  $\mathfrak{g}_t \cong \mathfrak{g}_s$  if and only if (3.12) has a rational solution such that  $x_{11}x_{22} - x_{12}x_{21} \neq 0$ . We state the following lemma in order to obtain less sophisticated conditions on  $s, t$ .

LEMMA 3.1. *Let  $s, t \in \mathbf{Q}$ . Then two conditions are equivalent:*  
i) *there exists a matrix*

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q})$$

such that  $x, y, z, w$  satisfy the system

$$(3.13) \quad \begin{cases} x^2 - (1 + s^2/4)z^2 = -y^2 + (1 + s^2/4)w^2 \neq 0, \\ 2(xy - (1 + s^2/4)zw) = t(x^2 - (1 + s^2/4)z^2). \end{cases}$$

ii) *there exists  $q \in \mathbf{Q}$  such that*

$$(3.14) \quad (t^2 + 4)(s^2 + 4) = q^2.$$

*Proof.* Let  $p = 1 + s^2/4$ ,  $r = 1 + t^2/4$ . The system (3.13) yields

$$(3.15) \quad \begin{cases} x^2 + y^2 = p(z^2 + w^2), \\ 2xy - tx^2 = p(2zw - tz^2). \end{cases}$$

After the change of variables

$$x = x_0, \quad y = \frac{1}{2}(y_0 + tx_0), \quad z = \frac{z_0}{p}, \quad w = \frac{1}{2}\left(w_0 + \frac{t}{p}z_0\right)$$

the system (3.15) can be rewritten as

$$(3.16) \quad \begin{cases} rx_0^2 + \frac{1}{4}y_0^2 = \frac{r}{p}z_0^2 + \frac{p}{4}w_0^2, \\ x_0y_0 = z_0w_0. \end{cases}$$

Geometrically, the system (3.16) defines the intersection  $I$  of two quadrics in the projective space  $\mathbf{P}^3$ . Let  $\sigma: \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$  be the Segre map. In homogeneous coordinates  $(a : b; \alpha : \beta)$  in  $\mathbf{P}^1 \times \mathbf{P}^1$ ,  $\sigma$  is defined by  $x_0 = a\alpha$ ,  $y_0 = b\beta$ ,  $z_0 = a\beta$ ,  $w_0 = b\alpha$ , and the image  $\sigma(\mathbf{P}^1 \times \mathbf{P}^1)$  is the zero locus of the polynomial  $x_0y_0 - z_0w_0$ .

It is not hard to verify that in coordinates  $(a : b; \alpha : \beta)$  the preimage  $\sigma^{-1}(I)$  is given by the following equation (corresponding to the first one of (3.16)):

$$(4ra^2 - pb^2)(p\alpha^2 - \beta^2) = 0.$$

Thus  $\sigma^{-1}(I)$  is the union of two pairs of lines (over  $\mathbf{R}$ ). The second pair defined by the equation  $p\alpha^2 - \beta^2 = 0$  yields  $xw - zy = \det(M) = 0$ . It follows that (3.15) has a rational solution if and only if the equation  $4ra^2 - pb^2 = 0$  has one, i.e.,  $p/4r$  is the square of a rational number. This is equivalent to (3.14). Note that the condition  $x^2 - pz^2 \neq 0$  in (3.13) is not very restrictive. This completes the proof of the lemma and of Theorem 1.

**COROLLARY 3.2.** *There are infinitely many non-isomorphic Lie algebras of the type  $\mathfrak{g}_s$  over  $\mathbf{Q}$ .*

*Proof.* Let  $s_1 = p_{11}$  be an odd prime. Consider  $s_1^2 + 4 = p_1^2 + 4 = p_{21}^{n_{21}} \dots$ . It is clear that  $s_1^2 + 4$  is not a square (this means that at least one of the  $n_{2j}$  is odd) and is not divisible by  $p_{11}$ , whence all the  $p_{2j} \neq p_{11}$ . Let  $s_2 = p_{11}p_{21} \dots$ . It follows that

$$s_2^2 + 4 = p_{31}^{n_{31}} \dots$$

is not a square and is not divisible by  $p_{ij}$  where  $i \leq 2$ . Then we set  $s_3 = p_{11}p_{21} \dots p_{31} \dots$  and so on.

In such a way we obtain an infinite sequence of numbers  $s_1, s_2 \dots$ . Let  $i < j$ . Note that  $(s_i^2 + 4)(s_j^2 + 4) \neq q^2$ ,  $q \in \mathbf{Q}$ . Indeed, by the construction  $(s_i^2 + 4)$  is divisible by some  $p$  and not divisible by  $p^2$ . Also,  $p$  divides  $s_j$ . Consequently, it does not divide  $s_j^2 + 4$ . This means that  $(s_i^2 + 4)(s_j^2 + 4)$  is divisible by  $p$  but not by  $p^2$ , and this completes the proof.