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skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis $B_1, \ldots, B_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over $\mathbf{Q}$ one can choose a canonical symplectic basis $\widehat{B_1}, \ldots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix $\widehat{M}$ representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis $B_1, \ldots, B_{d+1}$ (we omit the 'hats') of $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 Example of a free nilpotent algebra

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \ldots, x_n, \ldots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that $x_1, \ldots, x_n$ span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, $\mathfrak{h}$ is generated by $\{x_1, \ldots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi\colon \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of $\mathfrak{h}$ equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 More examples

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let $\mathfrak{g}_t$, $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \ldots, x_6\}$ such that

$$[x_1, x_2] = x_3\,, \quad [x_1, x_3] = t x_5\,, \quad [x_1, x_5] = x_6\,,$$
$$[x_2, x_3] = x_4\,, \quad [x_2, x_4] = x_5\,, \quad [x_3, x_4] = x_6\,,$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \ldots, x_6 \rangle$, $k = 2, \ldots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of $\mathfrak{g}_t$.
2. The centralizer $\mathfrak{C}$ of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by $x_2, \ldots, x_6$.

3. Real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over $\mathbf{Q}$.
5. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two Lie algebras with a unique rational form up to isomorphism.
6. Let $\mathfrak{g}$ be a split real simple Lie algebra of type G_2, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathfrak{n}_+$ is isomorphic to $\mathfrak{g}_0$.

3. Malcev's example

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a $\mathbf{Q}$-isomorphism between $\mathfrak{g}_t$ and $\mathfrak{g}_s$. It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6\rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6\rangle$ and the centralizer $\mathfrak{c}$ of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by $x_3, \ldots, x_6$.

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \ldots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \ldots \\ y_3 = \qquad\qquad a_{33}x_3 + a_{34}x_4 + \ldots \\ y_4 = \qquad\qquad\qquad a_{44}x_4 + \ldots \end{cases}$$

We do not explicit the expressions for y_5, y_6. Here $y_1, \ldots, y_6$ are basis elements of $\mathfrak{g}_s$ satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \ldots,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases} \tag{3.1}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned} \tag{3.2}$$