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Artikel: ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS

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skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis $B_1, \dots, B_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over $\mathbf{Q}$ one can choose a canonical symplectic basis $\widehat{B_1}, \dots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix $\widehat{M}$ representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis $B_1, \dots, B_{d+1}$ (we omit the 'hats') of $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 Example of a free nilpotent algebra

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that $x_1, \dots, x_n$ span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, $\mathfrak{h}$ is generated by $\{x_1, \dots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi\colon \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of $\mathfrak{h}$ equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 More examples

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let $\mathfrak{g}_t$, $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \dots, x_6\}$ such that

$$[x_1, x_2] = x_3\,, \quad [x_1, x_3] = t x_5\,, \quad [x_1, x_5] = x_6\,,$$
$$[x_2, x_3] = x_4\,, \quad [x_2, x_4] = x_5\,, \quad [x_3, x_4] = x_6\,,$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$, $k = 2, \dots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of $\mathfrak{g}_t$.
2. The centralizer $\mathfrak{C}$ of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by $x_2, \dots, x_6$.