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skew-symmetric  $d \times d$  matrix  $M = (\mu_{ij})$  with respect to the basis  $B_1, \dots, B_d$  ( $\text{mod } [\mathfrak{h}, \mathfrak{h}]$ ). Namely,  $[B_i, B_j] = \mu_{ij} B_{d+1}$ . Over  $\mathbf{Q}$  one can choose a canonical symplectic basis  $\widehat{B}_1, \dots, \widehat{B}_d$  ( $\text{mod } [\mathfrak{h}, \mathfrak{h}]$ ) so that the matrix  $\widehat{M}$  representing  $\omega$  has  $l$  blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of  $\omega$  is equal to  $2l$  and  $2l = d - m$ . In the basis  $B_1, \dots, B_{d+1}$  (we omit the ‘hats’) of  $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

## 2.2 EXAMPLE OF A FREE NILPOTENT ALGEBRA

Let  $\mathfrak{f}_c(n, \mathbf{R})$  be the free nilpotent Lie algebra of class  $c$  on  $n$  generators. Then  $\mathfrak{f}_c(n, \mathbf{R})$  has a unique rational form  $\mathfrak{f}_c(n, \mathbf{Q})$  up to isomorphism (cf. Theorem 2).

Indeed, let  $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$  be a rational form of  $\mathfrak{f}_c(n, \mathbf{R})$ . We may suppose that  $x_1, \dots, x_n$  span (modulo the derived subalgebra)  $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$ . Consequently,  $\mathfrak{h}$  is generated by  $\{x_1, \dots, x_n\}$  as a Lie algebra. There exists an epimorphism  $\pi: \mathfrak{f}_c(n, \mathbf{Q}) \rightarrow \mathfrak{h}$  because  $\mathfrak{f}_c(n, \mathbf{Q})$  is free. It must be an isomorphism since the dimension of  $\mathfrak{h}$  equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class  $c$  on  $n$  generators.

## 2.3 MORE EXAMPLES

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let  $\mathfrak{g}_t$ ,  $t \in \mathbf{R}$ , be a family of real 6-dimensional Lie algebras with a basis  $\{x_1, \dots, x_6\}$  such that

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= tx_5, & [x_1, x_5] &= x_6, \\ [x_2, x_3] &= x_4, & [x_2, x_4] &= x_5, & [x_3, x_4] &= x_6, \end{aligned}$$

other brackets being trivial. One can show that

1.  $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$ ,  $k = 2, \dots, 5$ , where  $C^k \mathfrak{g}_t$  are the terms of the lower central series of  $\mathfrak{g}_t$ .
2. The centralizer  $\mathfrak{C}$  of  $C^4 \mathfrak{g}_t$ , that is,  $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$  is spanned by  $x_2, \dots, x_6$ .