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<b>Autor:</b>	Semenov, Yu. S.
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**THEOREM 2.** *Let  $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$  be a free nilpotent Lie algebra of class  $c \geq 2$  on  $p$  generators. Then  $\mathfrak{g} \oplus \mathfrak{g}$  and  $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$  (regarded over  $\mathbf{R}$ ) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras  $\mathfrak{g}$  (two of them appear in Theorem 2 for  $p = c = 2$ ) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let  $G = UT_3(\mathbf{R})$  be the Lie group of upper triangular  $3 \times 3$ -matrices with 1 on the diagonal,  $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$  be Lie algebra of  $G$ . Consider  $G \times G$  and its Lie algebra  $\mathfrak{g} \oplus \mathfrak{g}$  which has infinitely many non-isomorphic rational forms  $\mathfrak{h}_m$  ( $m \geq 1$  is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let  $\Gamma_m$  and  $\Gamma_n$  be corresponding lattices in  $G \times G$  for distinct  $m, n$ . One can prove that the ratio of the covolumes of  $\Gamma_m$  and  $\Gamma_n$  with respect to a Haar measure on  $G \times G$  equals  $m\sqrt{m}/n\sqrt{n}$  up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3  $\Gamma_m$  and  $\Gamma_n$  are not commensurable in any sense.

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## 2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

### 2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra  $\mathfrak{g}$  has a  $\mathbf{Q}$ -form  $\mathfrak{h}$  and  $\mathfrak{i}$  (resp.  $\mathfrak{a}$ ) is an ideal (resp. a subalgebra) of  $\mathfrak{g}$ . We say that  $\mathfrak{i}$  (resp.  $\mathfrak{a}$ ) is rational if  $\mathfrak{i} \cap \mathfrak{h}$  (resp.  $\mathfrak{a} \cap \mathfrak{h}$ ) is a rational form of  $\mathfrak{i}$  (resp.  $\mathfrak{a}$ ). For instance, the terms  $C^k \mathfrak{g}$  of the lower central series of  $\mathfrak{g}$  are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that  $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$  is a rational form of the quotient Lie algebra  $\mathfrak{g}/\mathfrak{i}$ .

Let

$$(2.1) \quad \mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0$$

be a descending series of rational ideals of  $\mathfrak{g}$ . We say that a basis  $X = \{x_1, \dots, x_d\}$  of a rational form  $\mathfrak{h}$  is based on (2.1) if  $x_1, \dots, x_{p_1}$  generate  $\mathfrak{g} \bmod \mathfrak{i}_2$ ,  $x_1, \dots, x_{p_2}$  generate  $\mathfrak{g} \bmod \mathfrak{i}_3$  and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra  $\mathfrak{hei}_{2k+1}(\mathbf{R})$  has an  $\mathbf{R}$ -basis  $H_1, \dots, H_{2k+1}$  in which

$$(2.2) \quad [H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1},$$

other brackets being trivial. Here the 1-dimensional centre is spanned by  $H_{2k+1}$ .

Given an extension (1.1) one can attach to it a 2-cocycle  $\omega: \Lambda^2 \mathfrak{a} \rightarrow \mathbf{R}$  in the usual way. Also  $\omega$  can be regarded as a symplectic form on  $\mathfrak{a}$ . If  $\mathfrak{b} = \mathfrak{hei}_{2k+1}(\mathbf{R})$  then  $\omega$  is the canonical non-degenerate symplectic form with respect to the basis  $H_1, \dots, H_{2k} \pmod{\mathbf{R} \cdot H_{2k+1}}$ .

Let  $d = \dim_{\mathbf{R}} \mathfrak{a}$  and let  $m = d - \text{rank}(\omega)$  be the codimension of the kernel of  $\omega$ . It is not hard to see (cf. the proof of the proposition below) that the Lie algebra  $\mathfrak{b}$  is uniquely defined up to  $\mathbf{R}$ -isomorphism by  $d$  and  $m$ . Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m.$$

This implies that the centre of  $\mathfrak{b}$  is  $(m+1)$ -dimensional. Thus, two Lie algebras  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$  ( $\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2$ ) of type (1.1) are not isomorphic if  $m_1 \neq m_2$ .

Evidently,  $\mathfrak{hei}_{2k+1}(\mathbf{Q})$  is a rational form for  $\mathfrak{hei}_{2k+1}(\mathbf{R})$ .

The following proposition holds.

**PROPOSITION 2.1.** *In the above notation let  $\mathfrak{h}$  be a rational form of  $\mathfrak{b}$ . Let  $d = \dim(\mathfrak{b}) - 1$ ,  $m = \dim [\mathfrak{b}, \mathfrak{b}]$  and let  $\mathbf{Q}^m$  denote the abelian Lie  $\mathbf{Q}$ -algebra of dimension  $m$ . Then*

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

*over  $\mathbf{Q}$ , i.e., there is a unique rational form for  $\mathfrak{b}$  up to isomorphism.*

*Proof.* Choose a  $\mathbf{Q}$ -basis  $B_1, \dots, B_{d+1}$  for  $\mathfrak{h}$ . Either all brackets  $[B_i, B_j] = 0$ , and then  $\mathfrak{h} \cong \mathbf{Q}^{d+1}$ , or there are  $i, j$  such that  $[B_i, B_j] = C \neq 0$ .

We may suppose that  $C = B_{d+1}$ . Thus the derived subalgebra of  $\mathfrak{h}$  is spanned by  $B_{d+1}$ . The corresponding symplectic form  $\omega$  is represented by a

skew-symmetric  $d \times d$  matrix  $M = (\mu_{ij})$  with respect to the basis  $B_1, \dots, B_d$  ( $\bmod [\mathfrak{h}, \mathfrak{h}]$ ). Namely,  $[B_i, B_j] = \mu_{ij} B_{d+1}$ . Over  $\mathbf{Q}$  one can choose a canonical symplectic basis  $\widehat{B}_1, \dots, \widehat{B}_d$  ( $\bmod [\mathfrak{h}, \mathfrak{h}]$ ) so that the matrix  $\widehat{M}$  representing  $\omega$  has  $l$  blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of  $\omega$  is equal to  $2l$  and  $2l = d - m$ . In the basis  $B_1, \dots, B_{d+1}$  (we omit the ‘hats’) of  $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \dots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

## 2.2 EXAMPLE OF A FREE NILPOTENT ALGEBRA

Let  $\mathfrak{f}_c(n, \mathbf{R})$  be the free nilpotent Lie algebra of class  $c$  on  $n$  generators. Then  $\mathfrak{f}_c(n, \mathbf{R})$  has a unique rational form  $\mathfrak{f}_c(n, \mathbf{Q})$  up to isomorphism (cf. Theorem 2).

Indeed, let  $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$  be a rational form of  $\mathfrak{f}_c(n, \mathbf{R})$ . We may suppose that  $x_1, \dots, x_n$  span (modulo the derived subalgebra)  $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$ . Consequently,  $\mathfrak{h}$  is generated by  $\{x_1, \dots, x_n\}$  as a Lie algebra. There exists an epimorphism  $\pi: \mathfrak{f}_c(n, \mathbf{Q}) \rightarrow \mathfrak{h}$  because  $\mathfrak{f}_c(n, \mathbf{Q})$  is free. It must be an isomorphism since the dimension of  $\mathfrak{h}$  equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class  $c$  on  $n$  generators.

## 2.3 MORE EXAMPLES

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let  $\mathfrak{g}_t$ ,  $t \in \mathbf{R}$ , be a family of real 6-dimensional Lie algebras with a basis  $\{x_1, \dots, x_6\}$  such that

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= tx_5, & [x_1, x_5] &= x_6, \\ [x_2, x_3] &= x_4, & [x_2, x_4] &= x_5, & [x_3, x_4] &= x_6, \end{aligned}$$

other brackets being trivial. One can show that

1.  $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$ ,  $k = 2, \dots, 5$ , where  $C^k \mathfrak{g}_t$  are the terms of the lower central series of  $\mathfrak{g}_t$ .
2. The centralizer  $\mathfrak{C}$  of  $C^4 \mathfrak{g}_t$ , that is,  $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$  is spanned by  $x_2, \dots, x_6$ .

3. Real Lie algebras  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  are not isomorphic but  $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$ .
4. If  $t \in \mathbf{Q} \setminus \{0\}$  then the rational algebra  $\mathfrak{g}_t \cong \mathfrak{g}_1$  over  $\mathbf{Q}$ .
5.  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  are two Lie algebras with a unique rational form up to isomorphism.
6. Let  $\mathfrak{g}$  be a split real simple Lie algebra of type  $G_2$ ,  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  be the triangular decomposition of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Then  $\mathfrak{n}_+$  is isomorphic to  $\mathfrak{g}_0$ .

### 3. MALCEV'S EXAMPLE

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a  $\mathbf{Q}$ -isomorphism between  $\mathfrak{g}_t$  and  $\mathfrak{g}_s$ . It must be written in the following form (cf. [5]) since  $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$ ,  $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$  and the centralizer  $\mathfrak{c}$  of  $C^2\mathfrak{g}$ , which is an ideal in this case, is spanned by  $x_3, \dots, x_6$ .

$$\left\{ \begin{array}{l} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = \quad \quad \quad a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = \quad \quad \quad a_{44}x_4 + \dots \end{array} \right.$$

We do not explicit the expressions for  $y_5, y_6$ . Here  $y_1, \dots, y_6$  are basis elements of  $\mathfrak{g}_s$  satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$ . On the other hand,

$$\left\{ \begin{array}{l} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{array} \right.$$

Hence,

$$(3.1) \quad \left\{ \begin{array}{l} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{array} \right.$$

We need to compute the remaining two brackets. First of all,

$$(3.2) \quad \begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned}$$