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THEOREM 2. *Let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 2$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (regarded over \mathbf{R}) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras \mathfrak{g} (two of them appear in Theorem 2 for $p = c = 2$) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3 -matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G . Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms \mathfrak{h}_m ($m \geq 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n . One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

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2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra \mathfrak{g} has a \mathbf{Q} -form \mathfrak{h} and \mathfrak{i} (resp. \mathfrak{a}) is an ideal (resp. a subalgebra) of \mathfrak{g} . We say that \mathfrak{i} (resp. \mathfrak{a}) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of \mathfrak{i} (resp. \mathfrak{a}). For instance, the terms $C^k \mathfrak{g}$ of the lower central series of \mathfrak{g} are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.

Let

$$(2.1) \quad \mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0$$

be a descending series of rational ideals of \mathfrak{g} . We say that a basis $X = \{x_1, \dots, x_d\}$ of a rational form \mathfrak{h} is based on (2.1) if x_1, \dots, x_{p_1} generate $\mathfrak{g} \bmod \mathfrak{i}_2$, x_1, \dots, x_{p_2} generate $\mathfrak{g} \bmod \mathfrak{i}_3$ and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra $\mathfrak{hei}_{2k+1}(\mathbf{R})$ has an \mathbf{R} -basis H_1, \dots, H_{2k+1} in which

$$(2.2) \quad [H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1},$$

other brackets being trivial. Here the 1-dimensional centre is spanned by H_{2k+1} .

Given an extension (1.1) one can attach to it a 2-cocycle $\omega: \Lambda^2 \mathfrak{a} \rightarrow \mathbf{R}$ in the usual way. Also ω can be regarded as a symplectic form on \mathfrak{a} . If $\mathfrak{b} = \mathfrak{hei}_{2k+1}(\mathbf{R})$ then ω is the canonical non-degenerate symplectic form with respect to the basis $H_1, \dots, H_{2k} \pmod{\mathbf{R} \cdot H_{2k+1}}$.

Let $d = \dim_{\mathbf{R}} \mathfrak{a}$ and let $m = d - \text{rank}(\omega)$ be the codimension of the kernel of ω . It is not hard to see (cf. the proof of the proposition below) that the Lie algebra \mathfrak{b} is uniquely defined up to \mathbf{R} -isomorphism by d and m . Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m.$$

This implies that the centre of \mathfrak{b} is $(m+1)$ -dimensional. Thus, two Lie algebras \mathfrak{b}_1 and \mathfrak{b}_2 ($\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2$) of type (1.1) are not isomorphic if $m_1 \neq m_2$.

Evidently, $\mathfrak{hei}_{2k+1}(\mathbf{Q})$ is a rational form for $\mathfrak{hei}_{2k+1}(\mathbf{R})$.

The following proposition holds.

PROPOSITION 2.1. *In the above notation let \mathfrak{h} be a rational form of \mathfrak{b} . Let $d = \dim(\mathfrak{b}) - 1$, $m = \dim[\mathfrak{b}, \mathfrak{b}]$ and let \mathbf{Q}^m denote the abelian Lie \mathbf{Q} -algebra of dimension m . Then*

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

over \mathbf{Q} , i.e., there is a unique rational form for \mathfrak{b} up to isomorphism.

Proof. Choose a \mathbf{Q} -basis B_1, \dots, B_{d+1} for \mathfrak{h} . Either all brackets $[B_i, B_j] = 0$, and then $\mathfrak{h} \cong \mathbf{Q}^{d+1}$, or there are i, j such that $[B_i, B_j] = C \neq 0$.

We may suppose that $C = B_{d+1}$. Thus the derived subalgebra of \mathfrak{h} is spanned by B_{d+1} . The corresponding symplectic form ω is represented by a

skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis B_1, \dots, B_d (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over \mathbf{Q} one can choose a canonical symplectic basis $\widehat{B}_1, \dots, \widehat{B}_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix \widehat{M} representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis B_1, \dots, B_{d+1} (we omit the 'hats') of \mathfrak{h}

$$[B_1, B_2] = [B_3, B_4] = \dots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 EXAMPLE OF A FREE NILPOTENT ALGEBRA

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that x_1, \dots, x_n span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, \mathfrak{h} is generated by $\{x_1, \dots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi: \mathfrak{f}_c(n, \mathbf{Q}) \rightarrow \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of \mathfrak{h} equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 MORE EXAMPLES

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let \mathfrak{g}_t , $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \dots, x_6\}$ such that

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= tx_5, & [x_1, x_5] &= x_6, \\ [x_2, x_3] &= x_4, & [x_2, x_4] &= x_5, & [x_3, x_4] &= x_6, \end{aligned}$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$, $k = 2, \dots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of \mathfrak{g}_t .
2. The centralizer \mathfrak{C} of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by x_2, \dots, x_6 .

3. Real Lie algebras \mathfrak{g}_0 and \mathfrak{g}_1 are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over \mathbf{Q} .
5. \mathfrak{g}_0 and \mathfrak{g}_1 are two Lie algebras with a unique rational form up to isomorphism.
6. Let \mathfrak{g} be a split real simple Lie algebra of type G_2 , \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of \mathfrak{g} with respect to \mathfrak{h} . Then \mathfrak{n}_+ is isomorphic to \mathfrak{g}_0 .

3. MALCEV'S EXAMPLE

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a \mathbf{Q} -isomorphism between \mathfrak{g}_t and \mathfrak{g}_s . It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer \mathfrak{c} of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by x_3, \dots, x_6 .

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = a_{44}x_4 + \dots \end{cases}$$

We do not explicit the expressions for y_5, y_6 . Here y_1, \dots, y_6 are basis elements of \mathfrak{g}_s satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$(3.1) \quad \begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} (3.2) \quad [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned}$$