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ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS

by Yu. S. SEMENOV *)

ABSTRACT. We study the rational forms of real finite-dimensional nilpotent Lie algebras and the corresponding lattices in nilpotent Lie groups. We show that for some Lie algebras there are infinitely many such rational forms up to isomorphism and give a description of isomorphism classes in several 6-dimensional cases. Nilpotent Lie algebras with a unique rational form are also considered.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{R}$ and $\mathfrak{h}$ be a $\mathbf{Q}$-subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is a rational form (or $\mathbf{Q}$-form) of $\mathfrak{g}$ if there exists a basis X of $\mathfrak{h}$ over $\mathbf{Q}$ such that X is a basis of $\mathfrak{g}$ over $\mathbf{R}$. In other words, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ gives rise to an isomorphism $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{g}$.

In the sequel all Lie algebras are assumed to be nilpotent and finite-dimensional unless otherwise specified. The main purpose of the present work is to describe rational forms for some real nilpotent Lie algebras. The rational forms (or their isomorphism classes) in such algebras are closely related to lattices, i.e., discrete cocompact subgroups in nilpotent Lie groups.

Let G be a nilpotent connected 1-connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. It is well known that $\exp\colon \mathfrak{g} \to G$ and $\log\colon G \to \mathfrak{g}$ are two reciprocal diffeomorphisms. Let $\mathfrak{h}$ be a rational form of $\mathfrak{g}$ and $X = \{x_1, \dots, x_d\}$ be a basis of $\mathfrak{h}$. Malcev showed in [5] that the subgroup Γ of G generated by $\exp(rx_1), \dots, \exp(rx_d)$ (where r is an appropriate integer) is a lattice of G.

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If we change the basis of $\mathfrak{h}$ we can obtain another subgroup Γ_1 which is strictly commensurable to Γ. Here are three different definitions of commensurability of lattices.

DEFINITION 1. Two lattices Γ_1 and Γ_2 are strictly commensurable in a Lie group G if $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in both Γ_1 and Γ_2.

DEFINITION 2. Two lattices Γ_1 et Γ_2 are *commensurable* in a Lie group G if there is an element $g \in G$ such that Γ_1 and $g^{-1}\Gamma_2 g$ are strictly commensurable in G.

DEFINITION 3. Two lattices Γ_1 and Γ_2 are abstractly commensurable if there are two subgroups of finite index $H_i \leq \Gamma_i$ $(i = 1, 2)$ and an isomorphism $H_1 \cong H_2$.

We can deduce the following proposition from Malcev's results.

PROPOSITION 1.1. *There are three bijections:*

1. $\{\mathbf{Q}\text{-}\textit{forms of } \mathfrak{g}\} \cong \left\{\begin{matrix}\textit{Lattices } \Gamma \textit{ of } G \\ \textit{up to strict commensurability}\end{matrix}\right\};$

2. $\left\{\begin{matrix}\mathbf{Q}\text{-}\textit{forms of } \mathfrak{g} \\ \textit{up to adjoint automorphism } \mathrm{Ad}(g)\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{Lattices } \Gamma \textit{ of } G \\ \textit{up to commensurability}\end{matrix}\right\};$

3. $\left\{\begin{matrix}\mathbf{Q}\text{-}\textit{forms of } \mathfrak{g} \\ \textit{up to } \mathbf{Q}\text{-}\textit{isomorphism}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{Lattices } \Gamma \textit{ of } G \\ \textit{up to abstract commensurablity}\end{matrix}\right\}.$

Proof. The bijection 1 is a classical result of Malcev [5]. The bijection 2 follows immediately from the first one and the fact that the $\mathbf{Q}$-form $\mathrm{Ad}(g)\mathfrak{h}$ corresponds to the lattice $g\Gamma g^{-1}$ if and only if $\mathfrak{h}$ corresponds to Γ.

Let us prove the existence of the bijection 3. Let Γ_1 be a lattice abstractly commensurable with Γ_2 and $\mathfrak{h}_1$, $\mathfrak{h}_2$ be the corresponding $\mathbf{Q}$-forms. We may assume from the very beginning (possibly passing to the subgroups of finite index) that there is an isomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$. It is known [5] that α can be extended to an automorphism $\hat{\alpha}\colon G \to G$ and this automorphism gives rise to the automorphism $d\hat{\alpha}\colon \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. It is easy to see that $d\hat{\alpha}(\mathfrak{h}_1) = \mathfrak{h}_2$.

Conversely, let $\beta\colon \mathfrak{h}_1 \to \mathfrak{h}_2$ be an isomorphism of Lie algebras over $\mathbf{Q}$. It is clear that β can be considered as an $\mathbf{R}$-automorphism $\mathfrak{g} \to \mathfrak{g}$. It induces the automorphism $B\colon G \to G$ by means of the exponential map. Let $\Gamma_2 = B(\Gamma_1)$.

Then $\Gamma_1 \cong \Gamma_2$. Moreover, the $\mathbf{Q}$-form $\mathfrak{h}_2$ corresponds to the lattice Γ_2. Indeed, if $\exp(x) \in \Gamma_2$ then $\exp(x) = B(\exp(y)) = \exp(\beta(y))$ for some $y \in \mathfrak{h}_1$. Finally, $x = \beta(y) \in \mathfrak{h}_2$. $\square$

In this paper we are mostly interested in nilpotent Lie algebras with many rational forms (up to isomorphism). However, there are examples of nilpotent Lie algebras without rational forms. One of them, involving a Lie algebra of dimension 7 and class 6, is due to N. Bourbaki (see [1, Chap.1, §4, ex.18]). In fact, this is the minimal dimension for such an example. There are nilpotent Lie algebras of class 2 and of dimension $d \geq 10$ (see [5, 4]) without rational forms. Note that the corresponding Lie groups have no lattices at all.

It is a trivial exercise to show that every abelian Lie algebra has a unique rational form up to isomorphism. It follows from the results of Dixmier [3] that every real nilpotent Lie algebra $\mathfrak{g}$ of dimension ≤ 5 has the same property.

Consider the following *central* extension over $\mathbf{R}$ of an abelian finite dimensional real Lie algebra $\mathfrak{a}$:

$$0 \to \mathfrak{c} \to \mathfrak{b} \to \mathfrak{a} \to 0 \tag{1.1}$$

where $\mathfrak{c}$ is supposed to be a 1-dimensional ideal. The algebra $\mathfrak{b}$ is in fact the direct sum of a generalized Heisenberg algebra and an abelian one. In Section 2 we show that all such $\mathfrak{b}$ have a unique rational form (up to isomorphism) as well as all *free* real nilpotent Lie algebras.

In Section 3 we consider Malcev's example of a 6-dimensional Lie algebra of class 3 having infinitely many non-isomorphic rational forms in more details. Let $t \in \mathbf{R}$. Consider the nilpotent Lie algebra $\mathfrak{g}_t$ with a basis $x_1, \dots, x_6$ and the structure of Lie algebra given by the following relations:

$$\begin{gathered}[x_1, x_2] = x_4\,, \quad [x_1, x_3] = x_6\,, \quad [x_1, x_4] = x_5\,, \\ [x_2, x_3] = x_5 + tx_6\,, \quad [x_2, x_4] = x_6\,,\end{gathered} \tag{1.2}$$

other brackets being trivial. Malcev showed [5] that for all $t \in \mathbf{R}$ there is an isomorphism $\mathfrak{g}_t \cong \mathfrak{g}_0$ over $\mathbf{R}$ but, for instance, $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are not isomorphic over $\mathbf{Q}$.

The following theorem is proved.

THEOREM 1. *Let $s, t \in \mathbf{Q}$. The Lie algebras $\mathfrak{g}_s$ and $\mathfrak{g}_t$ are isomorphic over $\mathbf{Q}$ if and only if there is $q \in \mathbf{Q}$ such that $(s^2+4)(t^2+4) = q^2$.*

In Section 4 we propose a construction providing nilpotent Lie algebras with several non-isomorphic rational forms. As an application we prove the following theorem.

THEOREM 2. *Let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 2$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (regarded over $\mathbf{R}$) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras $\mathfrak{g}$ (two of them appear in Theorem 2 for $p = c = 2$) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3-matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G. Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms $\mathfrak{h}_m$ ($m \geq 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n. One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

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2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra $\mathfrak{g}$ has a $\mathbf{Q}$-form $\mathfrak{h}$ and $\mathfrak{i}$ (resp. $\mathfrak{a}$) is an ideal (resp. a subalgebra) of $\mathfrak{g}$. We say that $\mathfrak{i}$ (resp. $\mathfrak{a}$) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of $\mathfrak{i}$ (resp. $\mathfrak{a}$). For instance, the terms $C^k\mathfrak{g}$ of the lower central series of $\mathfrak{g}$ are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.