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ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS

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ABSTRACT. We study the rational forms of real finite-dimensional nilpotent Lie algebras and the corresponding lattices in nilpotent Lie groups. We show that for some Lie algebras there are infinitely many such rational forms up to isomorphism and give a description of isomorphism classes in several 6-dimensional cases. Nilpotent Lie algebras with a unique rational form are also considered.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{R}$ and $\mathfrak{h}$ be a $\mathbf{Q}$-subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is a rational form (or $\mathbf{Q}$-form) of $\mathfrak{g}$ if there exists a basis X of $\mathfrak{h}$ over $\mathbf{Q}$ such that X is a basis of $\mathfrak{g}$ over $\mathbf{R}$. In other words, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ gives rise to an isomorphism $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{g}$.

In the sequel all Lie algebras are assumed to be nilpotent and finite-dimensional unless otherwise specified. The main purpose of the present work is to describe rational forms for some real nilpotent Lie algebras. The rational forms (or their isomorphism classes) in such algebras are closely related to lattices, i.e., discrete cocompact subgroups in nilpotent Lie groups.

Let G be a nilpotent connected 1-connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. It is well known that $\exp\colon \mathfrak{g} \to G$ and $\log\colon G \to \mathfrak{g}$ are two reciprocal diffeomorphisms. Let $\mathfrak{h}$ be a rational form of $\mathfrak{g}$ and $X = \{x_1, \dots, x_d\}$ be a basis of $\mathfrak{h}$. Malcev showed in [5] that the subgroup Γ of G generated by $\exp(rx_1), \dots, \exp(rx_d)$ (where r is an appropriate integer) is a lattice of G.

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