

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COSET WEIGHT DISTRIBUTIONS OF CERTAIN BCH CODES
AND A FAMILY OF CURVES
Autor: van der Geer, G. / van der Vlugt, M.
Kapitel: §3. BOUNDS FOR $N(A,B)$
DOI: <https://doi.org/10.5169/seals-66065>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

(2.5) PROPOSITION. *Up to isogeny over $\mathbf{F}_{q=2^m}$ we have the splitting*

$$\text{Jac}(C/\tau) \sim \text{Jac}(C_{g_{\lambda^4}})^2 \times \text{Jac}(C_{g_{\lambda}}) \times E,$$

where $C_{g_{\lambda}}$ is as in (2.4) and E is the elliptic curve $y^2 + y = \lambda z^3$. The Prym variety P is isogenous to a product of six elliptic curves:

$$P \sim \text{Jac}(C_{f_1})^2 \times \text{Jac}(C_{f_3})^2 \times P',$$

where P' is a supersingular abelian surface whose trace of Frobenius $t(P')$ over \mathbf{F}_q satisfies

$$t(P') = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ -2(q-1) + 2t(C_{f_3}) & \text{if } m \text{ is even.} \end{cases}$$

Proof. The splitting of $\text{Jac}(C/\tau)$ follows directly from the description of C/τ as a fibre product and the splitting $\text{Jac}(C_{f_1+f_2}) \sim \text{Jac}(C_{g_{\lambda}}) \times E$ as obtained in (2.4). Furthermore, using Theorem (2.3) we see that

$$P \sim \text{Jac}(C_{f_1}) \times \text{Jac}(C_{f_2}) \times \text{Jac}(C_{f_3}) \times \text{Jac}(C_{f_1+f_2+f_3}).$$

We know $\text{Jac}(C_{f_1}) \cong \text{Jac}(C_{f_2})$ and that $\text{Jac}(C_{f_1+f_2+f_3})$ splits up to isogeny as $\text{Jac}(C_{f_3})$ and a 2-dimensional factor P' which is supersingular and up to isogeny a product of two elliptic curves. Using the map $x \mapsto w = x^3$ we see that $\#C_{f_1+f_2+f_3}(\mathbf{F}_q) = \#C_{f_3}(\mathbf{F}_q)$ if m is odd which implies $t(P') = 0$, while for m even

$$\#C_{f_1+f_2+f_3}(\mathbf{F}_q) - 2 = 3(\#C_{f_3}(\mathbf{F}_q) - 2).$$

This implies

$$t(C_{f_1+f_2+f_3}) - t(C_{f_3}) = -2(q-1) + 2t(C_{f_3})$$

and hence $t(P') = -2(q-1) + 2t(C_{f_3})$. This proves the assertion.

§ 3. BOUNDS FOR $N(A, B)$

Since the curve $C = C_{A,B}$ has genus 13 if $\lambda = A^2 + A + 1 + B \neq 0$ the Hasse-Weil-Serre bound for the number of \mathbf{F}_q -rational points $\#C_{A,B}(\mathbf{F}_q)$ says

$$(12) \quad q + 1 - 13[2\sqrt{q}] \leq \#C_{A,B}(\mathbf{F}_q) \leq q + 1 + 13[2\sqrt{q}].$$

The number $N(A, B)$ of S_4 -orbits of solutions of (1) with distinct $x_i \in \mathbf{F}_q$ satisfies

$$N(A, B) = (\#C_{A,B}(\mathbf{F}_q) - \text{contribution of } x = 0, 1, \infty)/24.$$

If $\text{Tr}(A + 1) = 0$ we have 12 rational points in the fibres above $0, 1, \infty$, while there are none if $\text{Tr}(A + 1) = 1$. Then (12) implies for $N(A, B)$ the inequalities

$$(q - 11 - 13[2\sqrt{q}])/24 \leq N(A, B) \leq (q + 1 + 13[2\sqrt{q}])/24.$$

By employing the decomposition of the Jacobian, especially Corollary (2.4), and taking into account that the possible values of the trace of Frobenius t of supersingular elliptic curves are $t = 0, \pm\sqrt{2q}$ for $q = 2^m$ with m odd we can refine these bounds and we obtain our main result on the numbers $N(A, B)$:

(3.1) THEOREM. For $q = 2^m$ with m odd the number $N(A, B)$ satisfies the following inequalities:

$$(13) \quad (q - 11 - 2\sqrt{2q} - 8[2\sqrt{q}])/24 \leq N(A, B) \leq (q + 1 + 2\sqrt{2q} + 8[2\sqrt{q}])/24.$$

Proof. The curve C_{f_1} is a supersingular elliptic curve, which implies that $-2\sqrt{2q} \leq 2t(C_{f_1}) \leq 2\sqrt{2q}$. Since the curve C_{f_3} has genus 1, $C_{f_1+f_3}$ has genus 2 and C_{g_λ} has genus 2 we obtain from the Hasse-Weil-Serre bound

$$-8[2\sqrt{q}] \leq 2t(C_{f_3}) + 2t(C_{f_1+f_3}) + t(C_{g_\lambda}) \leq 8[2\sqrt{q}].$$

Then it follows from Corollary (2.4) that the trace of Frobenius of $C_{A,B}$ is in the interval

$$[-2\sqrt{2q} - 8[2\sqrt{q}], 2\sqrt{2q} + 8[2\sqrt{q}]]$$

which yields (13).

In the following table we illustrate this by listing the intervals in which the numbers lie according to (13).

TABLE 1

q	32	128	512	2048	8192
interval	[0, 4]	[0, 14]	[4, 38]	[50, 120]	[270, 412]

For some further reflections on $N(A, B)$ we restrict our attention to the case $q = 2^m$ with m odd. The practice of searching for curves with many points tells us that it is highly improbable that in a fibre product of curves the traces of Frobenius of the individual components simultaneously reach their

maximal (or minimal) value. Hence it is very unlikely that the bounds given in (13) will be reached.

We intend to design an interval which contains almost all values of $N(A, B)$ using the description of C as a fibre product of the curves C_{f_i} for $i = 1, 2, 3$ given in (11) and a probabilistic argument on the distribution of traces of Frobenius.

The curves C_{f_1} and C_{f_2} are supersingular elliptic curves with the same trace of Frobenius $t = t(C_{f_i}) = 0, \pm\sqrt{2q}$. The curve $C_{f_1+f_2}$ has genus 3 which implies

$$-3[2\sqrt{q}] \leq t(C_{f_1+f_2}) \leq 3[2\sqrt{q}].$$

So the trace of Frobenius for the normalization of the fibre product $C_{f_1} \times_{\mathbf{P}^1} C_{f_2}$ satisfies

$$-3[2\sqrt{q}] - 2\sqrt{2q} \leq t \leq 3[2\sqrt{q}] + 2\sqrt{2q}.$$

We compute bounds for the number of $x \in \mathbf{P}^1 - \{0, 1, \infty\}$ above which we have 4 points in the fibre of $C_{f_1} \times_{\mathbf{P}^1} C_{f_2}$. If $\text{Tr}(A + 1) = 0$ we find in total 8 points above $x = 0, 1, \infty$, while we find none if $\text{Tr}(A + 1) = 1$. Subsequently we take into account that completely splitting $x \in \mathbf{P}^1 - \{0, 1, \infty\}$ occur in pairs $(x, 1/x)$ and we obtain the following proposition.

(3.2) PROPOSITION. *If we let*

$$M(f_1, f_2) = \frac{1}{2} \# \{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0, 1, \infty\} : x \text{ splits completely in } C_{f_1} \times_{\mathbf{P}^1} C_{f_2}\}$$

then we have for $\text{Tr}(A + 1) = 0$

$$\frac{q - 7 - 3[2\sqrt{q}] - 2\sqrt{2q}}{8} \leq M(f_1, f_2) \leq \frac{q - 7 + 3[2\sqrt{q}] + 2\sqrt{2q}}{8}$$

and for $\text{Tr}(A + 1) = 1$

$$\frac{q + 1 - 3[2\sqrt{q}] - 2\sqrt{2q}}{8} \leq M(f_1, f_2) \leq \frac{q + 1 + 3[2\sqrt{q}] + 2\sqrt{2q}}{8}.$$

We now consider the effect of the elliptic curve C_{f_3} in the fibre product. The j -invariant of C_{f_3} is $\lambda^{-4} \in \mathbf{F}_q^*$. This implies that $t(C_{f_3})$ is odd. For $\text{Tr}(A + 1) = 0$ we have $t(C_{f_3}) \equiv 1 \pmod{4}$ and there are 4 rational points together above $x = 0, 1, \infty$, while if $\text{Tr}(A + 1) = 1$ we have $t(C_{f_3}) \equiv 3 \pmod{4}$ and 2 rational points above $0, 1, \infty$. Furthermore, each element of \mathbf{F}_q^* occurs exactly once as j -invariant in the family of curves C_{f_3} . That implies that $t(C_{f_3})$ assumes each odd integer value in the interval $[-[2\sqrt{q}], [2\sqrt{q}]]$. So the number of completely splitting pairs assumes each integral value in the intervals mentioned in the following proposition.

(3.3) PROPOSITION. *If we let*

$$M(f_3) = \frac{1}{2} \# \{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0, 1, \infty\} : x \text{ splits completely in } C_{f_3}\}$$

then $M(f_3)$ assumes all integer values in the intervals

$$\begin{aligned} & \left[\frac{q-3-[2\sqrt{q}]}{4}, \frac{q-3+[2\sqrt{q}]}{4} \right] && \text{if } \text{Tr}(A+1) = 0, \\ & \left[\frac{q-1-[2\sqrt{q}]}{4}, \frac{q-1+[2\sqrt{q}]}{4} \right] && \text{if } \text{Tr}(A+1) = 1. \end{aligned}$$

Finally, we combine the two preceding propositions via a heuristic argument. Let

$$\begin{aligned} & M(f_1, f_2, f_3) \\ &= \frac{1}{2} \# \{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0, 1, \infty\} : x \text{ splits completely on } C_{f_1} \times_{\mathbf{P}^1} C_{f_2} \times_{\mathbf{P}^1} C_{f_3}\} \end{aligned}$$

Since there are $(q-2)/2$ pairs $(x, 1/x)$ ($x \neq 0, 1, \infty$) we expect

$$\begin{aligned} 2 \left(\frac{q-3-[2\sqrt{q}]}{4} \right) \left(\frac{q-7-3[2\sqrt{q}]-2\sqrt{2q}}{8} \right) / (q-2) &\leq M(f_1, f_2, f_3) \\ &\leq 2 \left(\frac{q-1+[2\sqrt{q}]}{4} \right) \left(\frac{q+1+3[2\sqrt{q}]+2\sqrt{2q}}{8} \right) / (q-2). \end{aligned}$$

If we work this out and neglect terms of order $1/\sqrt{q}$ and lower we find

$$\begin{aligned} (14) \quad \frac{q-4[2\sqrt{q}]-2\sqrt{2q}+4+4\sqrt{2}}{16} &\leq M(f_1, f_2, f_3) \\ &\leq \frac{q+4[2\sqrt{q}]+2\sqrt{2q}+14+4\sqrt{2}}{16}. \end{aligned}$$

Each completely splitting pair yields 16 solutions of (1) so to estimate the number of S_4 -orbits of solutions $N(A, B)$ we multiply the interval by $16/24$ to get an interval I . Since $N(A, B)$ is even we adapt the endpoints of the interval I just obtained slightly. Namely we consider the smallest interval with endpoints the positive even integers which contains I and we denote this interval by I^{even} .

(3.4) HEURISTICS. *The odds are that the values of $N(A, B)$ are in the interval*

$$I^{\text{even}} = \left[\frac{q-4[2\sqrt{q}]-2\sqrt{2q}+4\sqrt{2}+4}{24}, \frac{q+4[2\sqrt{q}]+2\sqrt{2q}+4\sqrt{2}+14}{24} \right]^{\text{even}}.$$

We illustrate this by a little table.