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PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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ABSTRACT. This paper is concerned with discrete versions of three well-known results from projective differential geometry: the four-vertex theorem, the theorem on six affine vertices, and Ghys' theorem on four zeroes of the Schwarzian derivative. We study the geometry of closed polygonal lines in $\mathbf{RP}^d$ and prove that polygons satisfying a certain convexity condition have at least $d+1$ flattenings. This result provides a new approach to the classical theorems mentioned.

1. INTRODUCTION

A *vertex* of a smooth plane curve is a point where the curve has 4^{th}-order contact with a circle (at a generic point the osculating circle has 3^{rd}-order contact with the curve). An *affine vertex* (or *sextactic point*) of a smooth plane curve is a point of 6^{th}-order contact with a conic. In 1909 S. Mukhopadhyaya [10] published two celebrated theorems, which are joined in the following statement:

Every closed smooth convex plane curve has at least 4 *distinct vertices and at least* 6 *distinct affine vertices.*

These results generated an extensive literature. From a modern point of view they are related, among other subjects, to the global singularity theory of wave fronts and to Sturm theory – see e.g. [1, 2, 4, 8, 17, 18] and references therein.