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## 9. THE POSITIVE CONE OF SOME PRODUCTS OF EVEN-DIMENSIONAL SPHERES

In this section, using known results from the theory of homotopy groups of spheres, we compute the positive cone of  $S^4 \times S^4$ ,  $S^4 \times S^6$ ,  $S^6 \times S^6$  and  $S^6 \times S^8$ . This computation will in particular show that the positive cone and the  $\gamma$ -cone do not coincide for  $S^4 \times S^4$ . Keeping notations as in Section 7, we describe the positive cone in terms of the geometric dimension function.

A) We start with the case of  $S^4 \times S^4$ .

**THEOREM 9.1.** *The geometric dimension on  $\tilde{K}(S^4 \times S^4)$  is given as follows: for  $x = ax_1 + bx_2 + lx_1x_2 \in \tilde{K}(S^4 \times S^4)$ , one has*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ 2 & \text{if } a \neq 0, b = l = 0 \\ 2 & \text{if } b \neq 0, l = ab/6, l \text{ even} \\ 3 & \text{if } b \neq 0, l = ab/6, l \text{ odd} \\ 4 & \text{if } l \neq ab/6 \end{cases}$$

*Proof.* Theorem 8.2 reduces the problem to the computation of the function  $s = s(ab)$ , i.e. to calculating  $\text{g-dim}(x)$  for the particular stable classes  $x = ax_1 + bx_2 + (ab/6)x_1x_2$  (where  $ab$  is a multiple of 6), or equivalently the order of  $[x_1, x_2]$  in both groups  $\pi_7(BU(3))$  and  $\pi_7(BU(2))$  (with a little abuse of notation, we denote both Whitehead products by the same symbol). By Samelson [Sam], one has

$$\pi_7(BU(2)) \cong \pi_6(U(2)) \cong \pi_6(SU(2)) \cong \pi_6(S^3) \cong \mathbf{Z}/12,$$

precisely generated by  $[x_1, x_2]$ . This shows that for these particular values of  $x$ ,  $\text{g-dim}(x) = 2$  if and only if  $ab$  is a multiple of 12. This completes the proof.  $\square$

**REMARK 9.2.**

i) Borel and Hirzebruch in [BoHi] (p.355), applying Bott's results of [Bott1], have proved that

$$\pi_{2n+1}(BU(n)) \cong \pi_{2n}(SU(n)) \cong \mathbf{Z}/n! \quad (n \geq 2),$$

hence  $\pi_7(BU(3)) \cong \mathbf{Z}/6$ . Moreover, Corollary 8.3 shows that the order of  $[x_1, x_2]$  in  $\pi_7(BU(3))$  is 6; it is consequently a generator.

ii) As already alluded to, we have just proved that  $S^4 \times S^4$  has its positive cone strictly contained in its  $\gamma$ -cone, although it is a torsion-free space.

B) As for  $S^4 \times S^4$ , classical results from the theory of homotopy groups of the unitary groups allow one to compute the positive cone of  $S^4 \times S^6$ . In this case, it coincides with the  $\gamma$ -cone.

THEOREM 9.3. *For the product  $S^4 \times S^6$ , one has*

$$K_+(S^4 \times S^6) = K_c(S^4 \times S^6) = K_\gamma(S^4 \times S^6).$$

*The latter is described in Theorem 7.1.*

*Proof.* By Lundell's tables [Lun] (see also [Mim]) and by Remark i) above, one has

$$\pi_9(BU(3)) \cong \mathbf{Z}/12 \quad \text{and} \quad \pi_9(BU(4)) \cong \mathbf{Z}/24.$$

Corollary 8.3 shows that  $[x_1, x_2]$  is of order 12 in  $\pi_9(BU(4))$ . By naturality of the Whitehead product, the homomorphism  $j_* = \pi_9(j)$ , induced by the map  $j: BU(3) \rightarrow BU(4)$ , takes the product  $[x_1, x_2] \in \pi_9(BU(3))$  to  $[x_1, x_2] \in \pi_9(BU(4))$ . This implies that  $[x_1, x_2]$  is of order 12 in  $\pi_9(BU(3))$  too, and that  $[ax_1, bx_2]$  vanishes in  $\pi_9(BU(3))$  precisely when it is zero in  $\pi_9(BU(4))$ . Together with Theorem 8.2, this completes the proof.  $\square$

REMARK 9.4. This proof shows in particular that  $[x_1, x_2]$  is a generator of  $\pi_9(BU(3)) \cong \mathbf{Z}/12$  and that the map  $j_*: \pi_9(BU(3)) \rightarrow \pi_9(BU(4))$  is injective.

C) By similar methods, we now show that the positive cone and the  $\gamma$ -cone coincide for  $S^6 \times S^6$  and then for  $S^6 \times S^8$ .

THEOREM 9.5. *For the product  $S^6 \times S^6$ , one has*

$$K_+(S^6 \times S^6) = K_c(S^6 \times S^6) = K_\gamma(S^6 \times S^6).$$

*The latter is given by Theorem 7.1.*

*Proof.* By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{11}(BU(3)) \cong \mathbf{Z}/30 \quad \text{and} \quad \pi_{11}(BU(5)) \cong \mathbf{Z}/120.$$

Corollary 8.3 shows that  $[x_1, x_2]$  is of order 30 in  $\pi_{11}(BU(5))$ . By naturality, the map  $j_* = \pi_{11}(j)$ , induced by  $j: BU(3) \rightarrow BU(5)$ , takes the Whitehead product  $[x_1, x_2] \in \pi_{11}(BU(3))$  to  $[x_1, x_2] \in \pi_{11}(BU(5))$ . This implies that  $[x_1, x_2]$  is of order 30 in  $\pi_{11}(BU(3))$  too, and that  $[ax_1, bx_2]$  vanishes in  $\pi_{11}(BU(3))$  precisely when it is zero in  $\pi_{11}(BU(5))$ . Together with Theorem 8.2, this completes the proof.  $\square$

REMARK 9.6.

i) This shows that  $[x_1, x_2]$  generates  $\pi_{11}(BU(3)) \cong \mathbf{Z}/30$  and that the map  $j_*: \pi_{11}(BU(3)) \rightarrow \pi_{11}(BU(5))$  is injective.

ii) We were also able to prove this theorem without appealing to results on homotopy groups of  $BU(n)$ . Using spectral sequence arguments, we have computed the first few stages of the Moore-Postnikov tower of the map  $BSU(3) \rightarrow BSU(5)$ . This computation, being extremely lengthy, is not given here (see [Matt]).

Now we move on to the product  $S^6 \times S^8$ .

THEOREM 9.7. *For the product  $S^6 \times S^8$ , one has*

$$K_+(S^6 \times S^8) = K_c(S^6 \times S^8) = K_\gamma(S^6 \times S^8).$$

*The latter is described in Theorem 7.1.*

*Proof.* By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{13}(BU(4)) \cong \mathbf{Z}/60 \quad \text{and} \quad \pi_{13}(BU(6)) \cong \mathbf{Z}/720.$$

Corollary 8.3 shows that  $[x_1, x_2]$  is of order 60 in  $\pi_{13}(BU(6))$ . By naturality, the map  $j_* = \pi_{13}(j)$ , induced by  $j: BU(4) \rightarrow BU(6)$ , takes the Whitehead product  $[x_1, x_2] \in \pi_{13}(BU(4))$  to  $[x_1, x_2] \in \pi_{13}(BU(6))$ . This implies that  $[x_1, x_2]$  is of order 60 in  $\pi_{13}(BU(4))$  too, and that  $[ax_1, bx_2]$  vanishes in  $\pi_{13}(BU(4))$  precisely when it is zero in  $\pi_{13}(BU(6))$ . Together with Theorem 8.2, this completes the proof.  $\square$

REMARK 9.8. The proof shows that  $[x_1, x_2]$  is a generator of the group  $\pi_{13}(BU(4)) \cong \mathbf{Z}/60$  and that the map  $j_*: \pi_{13}(BU(4)) \rightarrow \pi_{13}(BU(6))$  is injective.

## 10. "GAPS IN COHOMOLOGY" AND THE $\gamma$ -CONE

In the present section, we are interested in spaces having a "gap in cohomology", more precisely we look at spaces obtained by attaching a single large-dimensional cell to a finite CW-complex  $Y$ . For such spaces, the integral cohomology is zero between the dimension of  $Y$  and the top-dimensional class. The products  $S^n \times S^m$  are typical examples (see Section 8). For this kind of spaces, the  $c$ -cone obviously cannot give information in the