9. The positive cone of some products of even-dimensional spheres

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9. The positive cone of some products of even-dimensional spheres

In this section, using known results from the theory of homotopy groups of spheres, we compute the positive cone of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This computation will in particular show that the positive cone and the γ-cone do not coincide for $S^4 \times S^4$. Keeping notations as in Section 7, we describe the positive cone in terms of the geometric dimension function.

A) We start with the case of $S^4 \times S^4$.

THEOREM 9.1. *The geometric dimension on $\widetilde{K}(S^4 \times S^4)$ is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^4 \times S^4)$, one has*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ 2 & \text{if } a \neq 0,\ b = l = 0 \\ 2 & \text{if } b \neq 0,\ l = ab/6,\ l \text{ even} \\ 3 & \text{if } b \neq 0,\ l = ab/6,\ l \text{ odd} \\ 4 & \text{if } l \neq ab/6 \end{cases}$$

Proof. Theorem 8.2 reduces the problem to the computation of the function $s = s(ab)$, i.e. to calculating g-dim(x) for the particular stable classes $x = ax_1 + bx_2 + (ab/6)x_1x_2$ (where ab is a multiple of 6), or equivalently the order of $[x_1, x_2]$ in both groups $\pi_7(BU(3))$ and $\pi_7(BU(2))$ (with a little abuse of notation, we denote both Whitehead products by the same symbol). By Samelson [Sam], one has

$$\pi_7(BU(2)) \cong \pi_6(U(2)) \cong \pi_6(SU(2)) \cong \pi_6(S^3) \cong \mathbf{Z}/12\,,$$

precisely generated by $[x_1, x_2]$. This shows that for these particular values of x, g-dim$(x) = 2$ if and only if ab is a multiple of 12. This completes the proof. □

REMARK 9.2.

i) Borel and Hirzebruch in [BoHi] (p. 355), applying Bott's results of [Bott1], have proved that

$$\pi_{2n+1}(BU(n)) \cong \pi_{2n}(SU(n)) \cong \mathbf{Z}/n! \quad (n \geq 2),$$

hence $\pi_7(BU(3)) \cong \mathbf{Z}/6$. Moreover, Corollary 8.3 shows that the order of $[x_1, x_2]$ in $\pi_7(BU(3))$ is 6; it is consequently a generator.

ii) As already alluded to, we have just proved that $S^4 \times S^4$ has its positive cone strictly contained in its γ-cone, although it is a torsion-free space.

B) As for $S^4 \times S^4$, classical results from the theory of homotopy groups of the unitary groups allow one to compute the positive cone of $S^4 \times S^6$. In this case, it coincides with the γ-cone.

THEOREM 9.3. *For the product $S^4 \times S^6$, one has*

$$K_+(S^4 \times S^6) = K_c(S^4 \times S^6) = K_\gamma(S^4 \times S^6).$$

The latter is described in Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]) and by Remark i) above, one has

$$\pi_9(BU(3)) \cong \mathbf{Z}/12 \quad \text{and} \quad \pi_9(BU(4)) \cong \mathbf{Z}/24.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(4))$. By naturality of the Whitehead product, the homomorphism $j_* = \pi_9(j)$, induced by the map $j\colon BU(3) \longrightarrow BU(4)$, takes the product $[x_1, x_2] \in \pi_9(BU(3))$ to $[x_1, x_2] \in \pi_9(BU(4))$. This implies that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_9(BU(3))$ precisely when it is zero in $\pi_9(BU(4))$. Together with Theorem 8.2, this completes the proof. $\square$

REMARK 9.4. This proof shows in particular that $[x_1, x_2]$ is a generator of $\pi_9(BU(3)) \cong \mathbf{Z}/12$ and that the map $j_*\colon \pi_9(BU(3)) \longrightarrow \pi_9(BU(4))$ is injective.

C) By similar methods, we now show that the positive cone and the γ-cone coincide for $S^6 \times S^6$ and then for $S^6 \times S^8$.

THEOREM 9.5. *For the product $S^6 \times S^6$, one has*

$$K_+(S^6 \times S^6) = K_c(S^6 \times S^6) = K_\gamma(S^6 \times S^6).$$

The latter is given by Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{11}(BU(3)) \cong \mathbf{Z}/30 \quad \text{and} \quad \pi_{11}(BU(5)) \cong \mathbf{Z}/120.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(5))$. By naturality, the map $j_* = \pi_{11}(j)$, induced by $j\colon BU(3) \longrightarrow BU(5)$, takes the Whitehead product $[x_1, x_2] \in \pi_{11}(BU(3))$ to $[x_1, x_2] \in \pi_{11}(BU(5))$. This implies that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{11}(BU(3))$ precisely when it is zero in $\pi_{11}(BU(5))$. Together with Theorem 8.2, this completes the proof. $\square$

REMARK 9.6.

i) This shows that $[x_1, x_2]$ generates $\pi_{11}(BU(3)) \cong \mathbf{Z}/30$ and that the map $j_*\colon \pi_{11}(BU(3)) \longrightarrow \pi_{11}(BU(5))$ is injective.

ii) We were also able to prove this theorem without appealing to results on homotopy groups of $BU(n)$. Using spectral sequence arguments, we have computed the first few stages of the Moore-Postnikov tower of the map $BSU(3) \longrightarrow BSU(5)$. This computation, being extremely lengthy, is not given here (see [Matt]).

Now we move on to the product $S^6 \times S^8$.

THEOREM 9.7. *For the product* $S^6 \times S^8$, *one has*

$$K_+(S^6 \times S^8) = K_c(S^6 \times S^8) = K_\gamma(S^6 \times S^8).$$

The latter is described in Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{13}(BU(4)) \cong \mathbf{Z}/60 \quad \text{and} \quad \pi_{13}(BU(6)) \cong \mathbf{Z}/720.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(6))$. By naturality, the map $j_* = \pi_{13}(j)$, induced by $j\colon BU(4) \longrightarrow BU(6)$, takes the Whitehead product $[x_1, x_2] \in \pi_{13}(BU(4))$ to $[x_1, x_2] \in \pi_{13}(BU(6))$. This implies that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(4))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{13}(BU(4))$ precisely when it is zero in $\pi_{13}(BU(6))$. Together with Theorem 8.2, this completes the proof. $\square$

REMARK 9.8. The proof shows that $[x_1, x_2]$ is a generator of the group $\pi_{13}(BU(4)) \cong \mathbf{Z}/60$ and that the map $j_*\colon \pi_{13}(BU(4)) \longrightarrow \pi_{13}(BU(6))$ is injective.

10. "GAPS IN COHOMOLOGY" AND THE γ-CONE

In the present section, we are interested in spaces having a "gap in cohomology", more precisely we look at spaces obtained by attaching a single large-dimensional cell to a finite CW-complex Y. For such spaces, the integral cohomology is zero between the dimension of Y and the top-dimensional class. The products $S^n \times S^m$ are typical examples (see Section 8). For this kind of spaces, the c-cone obviously cannot give information in the