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We now turn to the product  $S^{2n-1} \times S^{2m-1}$ . From the six-term exact sequence of the pair  $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$ , with quotient the smash product  $S^{2n-1} \wedge S^{2m-1}$  homeomorphic to  $S^{2m+2n-2}$ , we get an isomorphism

$$q^*: \widetilde{K}(S^{2m+2n-2}) \longrightarrow \widetilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map  $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ . By Theorem 4.1, the space  $Y = S^{2n+2m-2}$  satisfies the hypothesis of Proposition 5.5 and we deduce the

**THEOREM 6.2.** *The map  $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$  induces an isomorphism of positive cones, and, for  $S^{2n-1} \times S^{2m-1}$ , the  $\gamma$ -cone and the  $c$ -cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \xrightarrow{q^*} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

**REMARK 6.3.** According to Blackadar ([Bla2], 6.10.2), the positive cone of the  $n$ -torus  $(S^1)^n$  has been partially computed by Villadsen.

## 7. THE $\gamma$ -CONE OF $S^{2n} \times S^{2m}$ AND THE POSITIVE CONE OF $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd-dimensional sphere by any sphere, whereas the case of a product of two even-dimensional spheres is much more involved. On the other hand, the  $\gamma$ -cone of such a product is in the scope of the present notes. We perform this calculation by computing the  $c$ -cone and appealing to Proposition 3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}), \quad \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where  $p$  and  $q$  are the projections onto the factors. Writing  $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x_1$  and  $\widetilde{K}(S^{2m}) = \mathbf{Z} \cdot x_2$ , and letting  $y_1 := p^*(x_1)$  and  $y_2 := q^*(x_2)$ , we deduce that

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot y_1 \oplus \mathbf{Z} \cdot y_2 \oplus \mathbf{Z} \cdot y_1 y_2.$$

The product structure on  $\widetilde{K}(S^{2n} \times S^{2m})$  is given by  $y_1^2 = 0$  and  $y_2^2 = 0$ . One has  $y_1 y_2 = \pi^*(y)$ , where  $\pi: S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$  and  $y$  is a suitable generator of  $\widetilde{K}(S^{2n+2m})$ . Let  $i: S^{2n} \hookrightarrow S^{2n} \times S^{2m}$  and  $j: S^{2m} \hookrightarrow S^{2n} \times S^{2m}$  be the inclusions. One has  $i^*(y_1) = x_1$  and  $j^*(y_2) = x_2$ , and (by Theorem 4.1 and a double application of Proposition 5.1), for any  $k \in \mathbf{Z} \setminus \{0\}$ , one has

$g\text{-dim}(ky_1) = g\text{-dim}(kx_1) = n$ ; similarly  $g\text{-dim}(ky_2) = g\text{-dim}(kx_2) = m$ . This justifies that, from now on, we write  $x_1$  and  $x_2$  for  $y_1$  and  $y_2$  respectively.

Let  $a_1 \in H^{2n}(S^{2n}; \mathbf{Z})$  and  $a_2 \in H^{2m}(S^{2m}; \mathbf{Z})$  be suitable generators (referring to Proposition 2.4). As before, it is justified to write

$$\widetilde{H}^*(S^{2n} \times S^{2m}; \mathbf{Z}) = \mathbf{Z} \cdot a_1 \oplus \mathbf{Z} \cdot a_2 \oplus \mathbf{Z} \cdot a_1 a_2.$$

Let us assume  $n \leq m$ . Consider an element  $x = ax_1 + bx_2 + lx_1 x_2$  in the group  $\widetilde{K}(S^{2n} \times S^{2m})$ . For the Chern classes, invoking Proposition 2.4 and “exponentiality” of the total Chern class, we compute

$$\begin{aligned} c(x) &= c(ax_1)c(bx_2)c(lx_1 x_2) \\ &= 1 + (-1)^{n-1}a(n-1)! \cdot a_1 + (-1)^{m-1}b(m-1)! \cdot a_2 \\ &\quad + (-1)^{n+m}(ab(n-1)!(m-1)! - l(n+m-1)!) \cdot a_1 a_2. \end{aligned}$$

This immediately gives the  $\gamma$ -cone (which coincides with the  $c$ -cone) in terms of the  $\gamma$ -dimension function.

**THEOREM 7.1.** *For  $n \leq m$ , the  $\gamma$ -dimension on  $\widetilde{K}(S^{2n} \times S^{2m})$  is given as follows: for  $x = ax_1 + bx_2 + lx_1 x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$ , one has*

$$\gamma\text{-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0, b = l = 0 \\ m & \text{if } b \neq 0, l = ab(n-1)!(m-1)/(n+m-1)! \\ n+m & \text{if } l \neq ab(n-1)!(m-1)/(n+m-1)! \end{cases}$$

Moreover, for  $k \neq 0$ , one has

$$g\text{-dim}(kx_1) = n \quad \text{and} \quad g\text{-dim}(kx_2) = m.$$

This theorem allows us to give some interesting information on the positive cone of the product  $S^{2n} \times S^{2m}$ . We will state the result as Theorem 8.2 in the following section, because the tools developed there allow one to make a crucial improvement.

Combined with Theorem 2.3, Theorem 7.1 enables one to compute completely the positive cone of  $S^2 \times S^{2n}$ .

**THEOREM 7.2.** *For the product  $S^2 \times S^{2n}$ , we have*

$$K_+(S^2 \times S^{2n}) = K_c(S^2 \times S^{2n}) = K_\gamma(S^2 \times S^{2n}).$$

*The latter is given by Theorem 7.1.*