

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES
Autor: MATTHEY, Michel / SUTER, Ulrich
Kapitel: 7. The -cone of $S^{2n} \times S^{2m}$ and the positive cone of $S^2 \times S^{2n}$
DOI: <https://doi.org/10.5169/seals-65432>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

We now turn to the product $S^{2n-1} \times S^{2m-1}$. From the six-term exact sequence of the pair $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$, with quotient the smash product $S^{2n-1} \wedge S^{2m-1}$ homeomorphic to $S^{2m+2n-2}$, we get an isomorphism

$$q^*: \tilde{K}(S^{2m+2n-2}) \longrightarrow \tilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$. By Theorem 4.1, the space $Y = S^{2m+2n-2}$ satisfies the hypothesis of Proposition 5.5 and we deduce the

THEOREM 6.2. *The map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ induces an isomorphism of positive cones, and, for $S^{2n-1} \times S^{2m-1}$, the γ -cone and the c -cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \stackrel{q^*}{\cong} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

REMARK 6.3. According to Blackadar ([Bla2], 6.10.2), the positive cone of the n -torus $(S^1)^n$ has been partially computed by Villadsen.

7. THE γ -CONE OF $S^{2n} \times S^{2m}$ AND THE POSITIVE CONE OF $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd-dimensional sphere by any sphere, whereas the case of a product of two even-dimensional spheres is much more involved. On the other hand, the γ -cone of such a product is in the scope of the present notes. We perform this calculation by computing the c -cone and appealing to Proposition 3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}), \quad \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where p and q are the projections onto the factors. Writing $\tilde{K}(S^{2n}) = \mathbf{Z} \cdot x_1$ and $\tilde{K}(S^{2m}) = \mathbf{Z} \cdot x_2$, and letting $y_1 := p^*(x_1)$ and $y_2 := q^*(x_2)$, we deduce that

$$\tilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot y_1 \oplus \mathbf{Z} \cdot y_2 \oplus \mathbf{Z} \cdot y_1 y_2.$$

The product structure on $\tilde{K}(S^{2n} \times S^{2m})$ is given by $y_1^2 = 0$ and $y_2^2 = 0$. One has $y_1 y_2 = \pi^*(y)$, where $\pi: S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ and y is a suitable generator of $\tilde{K}(S^{2n+2m})$. Let $i: S^{2n} \hookrightarrow S^{2n} \times S^{2m}$ and $j: S^{2m} \hookrightarrow S^{2n} \times S^{2m}$ be the inclusions. One has $i^*(y_1) = x_1$ and $j^*(y_2) = x_2$, and (by Theorem 4.1 and a double application of Proposition 5.1), for any $k \in \mathbf{Z} \setminus \{0\}$, one has

$\text{g-dim}(ky_1) = \text{g-dim}(kx_1) = n$; similarly $\text{g-dim}(ky_2) = \text{g-dim}(kx_2) = m$. This justifies that, from now on, we write x_1 and x_2 for y_1 and y_2 respectively.

Let $a_1 \in H^{2n}(S^{2n}; \mathbf{Z})$ and $a_2 \in H^{2m}(S^{2m}; \mathbf{Z})$ be suitable generators (referring to Proposition 2.4). As before, it is justified to write

$$\tilde{H}^*(S^{2n} \times S^{2m}; \mathbf{Z}) = \mathbf{Z} \cdot a_1 \oplus \mathbf{Z} \cdot a_2 \oplus \mathbf{Z} \cdot a_1 a_2.$$

Let us assume $n \leq m$. Consider an element $x = ax_1 + bx_2 + lx_1x_2$ in the group $\tilde{K}(S^{2n} \times S^{2m})$. For the Chern classes, invoking Proposition 2.4 and “exponentiality” of the total Chern class, we compute

$$\begin{aligned} c(x) &= c(ax_1)c(bx_2)c(lx_1x_2) \\ &= 1 + (-1)^{n-1}a(n-1)! \cdot a_1 + (-1)^{m-1}b(m-1)! \cdot a_2 \\ &\quad + (-1)^{n+m}(ab(n-1)!(m-1)! - l(n+m-1)!) \cdot a_1 a_2. \end{aligned}$$

This immediately gives the γ -cone (which coincides with the c -cone) in terms of the γ -dimension function.

THEOREM 7.1. *For $n \leq m$, the γ -dimension on $\tilde{K}(S^{2n} \times S^{2m})$ is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \tilde{K}(S^{2n} \times S^{2m})$, one has*

$$\gamma\text{-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0, b = l = 0 \\ m & \text{if } b \neq 0, l = ab(n-1)!(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab(n-1)!(m-1)!/(n+m-1)! \end{cases}$$

Moreover, for $k \neq 0$, one has

$$\text{g-dim}(kx_1) = n \quad \text{and} \quad \text{g-dim}(kx_2) = m.$$

This theorem allows us to give some interesting information on the positive cone of the product $S^{2n} \times S^{2m}$. We will state the result as Theorem 8.2 in the following section, because the tools developed there allow one to make a crucial improvement.

Combined with Theorem 2.3, Theorem 7.1 enables one to compute completely the positive cone of $S^2 \times S^{2n}$.

THEOREM 7.2. *For the product $S^2 \times S^{2n}$, we have*

$$K_+(S^2 \times S^{2n}) = K_c(S^2 \times S^{2n}) = K_\gamma(S^2 \times S^{2n}).$$

The latter is given by Theorem 7.1.