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LEMMA 4.2. For  $m \geq 2k + 1$ , the group  $\pi_m(BU(k))$  is finite.

*Proof.* We fix  $m \geq 3$ . The fibration  $BU(k-1) \rightarrow BU(k)$ , with fibre  $S^{2k-1}$ , yields the following long exact sequence in homotopy:

$$\dots \rightarrow \pi_m(S^{2k-1}) \rightarrow \pi_m(BU(k-1)) \rightarrow \pi_m(BU(k)) \rightarrow \pi_{m-1}(S^{2k-1}) \rightarrow \dots$$

By Serre [Serre],  $\pi_j(S^{2k-1})$  is finite for  $j \neq 2k-1$ , and we can conclude by induction over  $k$  (with  $k \geq 1$  and  $2k+1 \leq m$ ), since when  $k=1$ , one has  $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$  for  $m \geq 3$ .  $\square$

From this, we now infer that the image of  $(i_k)_*$  is zero for  $k < n$ . This implies that  $\text{g-dim}(lx) = n$  when  $l \neq 0$ , and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital  $C^*$ -algebras of type AF by means of their  $K$ -theory, their positive cone and the  $K$ -theory class [1] of the unit (see [Bla1]), it is important to single out the fact that the positive cone of  $S^{2n}$  and that of  $S^{2m}$  are non-isomorphic as monoids if  $n$  is different from  $m$ . (There is no need here to distinguish the  $K$ -theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For  $n \geq 1$ , let  $M_n$  denote the positive cone of  $S^{2n}$  (identified as above with a sub-monoid of  $\mathbf{Z}^2$ , in order to designate its elements). The abelian monoid  $M_n$  has a minimal set  $A_n$  of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0, 1)\} \cup \{(k, n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function  $\sigma: A_n \rightarrow \{2, 3, \dots\}$  defined, for  $x \in A_n$ , by

$$\sigma(x) := \min \{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}.$$

Clearly, such an  $l$  exists for any  $x \in A_n$  and  $\sigma(A_n) = \{2, 2n\}$ . Since  $A_n$  and  $\sigma$  are isomorphism invariants of  $M_n$ , this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words,  $K(S^{2n-1}) = \mathbf{Z}$  and  $K_+(S^{2n-1}) = \mathbf{N}$ .

## 5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the  $\gamma$ -cone and the  $c$ -cone.

The following result is obvious.

PROPOSITION 5.1. Let  $f: X \rightarrow Y$  be a map between connected finite CW-complexes. Let  $f^*: K(Y) \rightarrow K(X)$  be the  $\lambda$ -homomorphism induced by  $f$ . Then, for any  $y \in \tilde{K}(Y)$ , one has

$$\begin{aligned} \text{g-dim}(f^*(y)) &\leq \text{g-dim}(y) \\ \gamma\text{-dim}(f^*(y)) &\leq \gamma\text{-dim}(y) \\ \text{c-dim}(f^*(y)) &\leq \text{c-dim}(y), \end{aligned}$$

and in particular,

$$\begin{aligned} f^*(K_+(Y)) &\subseteq K_+(X) \\ f^*(K_\gamma(Y)) &\subseteq K_\gamma(X) \\ f^*(K_c(Y)) &\subseteq K_c(X). \end{aligned}$$

Furthermore, if  $f^*$  is an isomorphism, then

$$f^*(K_\gamma(Y)) = K_\gamma(X).$$

For the next corollary we need a new definition.

DEFINITION 5.2. Let  $X$  and  $Y$  be two connected finite CW-complexes. A map  $f: X \rightarrow Y$  is called a  $K^0$ -equivalence (or  $K$ -equivalence for short) if there exists a map  $g: Y \rightarrow X$  such that, on the level of the  $K^0$ -groups,

$$f^* \circ g^* = \text{Id}_{K^0(X)} \quad \text{and} \quad g^* \circ f^* = \text{Id}_{K^0(Y)}.$$

Note that a  $K$ -equivalence is *not* necessarily a homotopy equivalence: there are homotopically non-trivial (i.e. non-contractible) finite CW-complexes  $X$  for which  $\tilde{K}(X) = 0 = \tilde{K}(pt)$ ; see example iii) below.

PROPOSITION 5.3. If  $f: X \rightarrow Y$  is a  $K$ -equivalence, then  $f$  induces the following isomorphisms of semigroups:

$$K_+(Y) \xrightarrow{f^*} K_+(X) \quad \text{and} \quad K_\gamma(Y) \xrightarrow{f^*} K_\gamma(X).$$

*Proof.* Applying Proposition 5.1 twice, we get (in the notations of Definition 5.2)

$$K_+(X) = f^* \circ g^*(K_+(X)) \subseteq f^*(K_+(Y)) \subseteq K_+(X).$$

This establishes the first isomorphism, whereas the second one is obvious.  $\square$

The following result is more technical to state.

COROLLARY 5.4. *Let  $X$  and  $Y$  be two connected finite CW-complexes. Assume that  $K^1(X) = 0$  and that  $\tilde{K}^0(Y) = 0$ . Then the projection  $p: X \times Y \rightarrow X$  induces isomorphisms*

$$K_+(X) \stackrel{p^*}{\cong} K_+(X \times Y) \quad \text{and} \quad K_\gamma(X) \stackrel{p^*}{\cong} K_\gamma(X \times Y).$$

*Proof.* Invoking the Künneth theorem for  $K$ -theory, our hypotheses imply that  $p^*: K^0(X) \rightarrow K^0(X \times Y)$  is an isomorphism with inverse  $i^*$ , where  $i$  is the inclusion of  $X$  in  $X \times Y$ . Consequently,  $p^*$  is a  $K$ -equivalence.  $\square$

The following is a useful result.

PROPOSITION 5.5. *Let  $X$  and  $Y$  be connected finite CW-complexes. Assume that the positive cone and the  $\gamma$ -cone of  $Y$  coincide, and let  $f: X \rightarrow Y$  be a map inducing an isomorphism  $f^*: K(Y) \rightarrow K(X)$ . Then  $f$  induces an isomorphism of positive cones, and the  $\gamma$ -cone of  $X$  coincides with the positive cone:*

$$K_+(Y) \stackrel{f^*}{\cong} K_+(X) = K_\gamma(X).$$

*Proof.* By Proposition 5.1 we have  $f^*(K_+(Y)) = f^*(K_\gamma(Y)) = K_\gamma(X)$  and  $f^*(K_+(Y)) \subseteq K_+(X)$ , hence  $K_\gamma(X) \subseteq K_+(X)$ . We conclude with iii) of Proposition 3.2.  $\square$

#### EXAMPLES.

i) Let  $X$  be a connected finite CW-complex of dimension  $\leq 3$ . Since for suitable CW-decompositions, one has  $BU(1)^{[3]} = BU^{[3]}$  and since  $BU(1) = CP^\infty = K(\mathbf{Z}, 2)$ , any  $x \in \tilde{K}(X) = [X, BU]$  lifts to a class in  $[X, BU(1)]$ , giving an isomorphism  $\tilde{K}(X) \cong H^2(X; \mathbf{Z})$  mapping  $x$  to  $c_1(x)$ . It follows that the positive cone coincides with the  $c$ -cone and is given by

$$K_+(X) = \mathbf{N} \times \{0\} \cup \mathbf{N}^* \times \tilde{K}(X) \subset \mathbf{Z} \times \tilde{K}(X).$$

ii) Example i) applies to a closed oriented surface  $\Sigma_g$  of genus  $g$ . Since it is torsion-free, its positive cone coincides with its  $c$ -cone and with its  $\gamma$ -cone. Moreover, let  $f: \Sigma_g \rightarrow S^2$  be a map of degree 1 (it exists, since both the 2-sphere and  $\Sigma_g$  are quotients of the square  $[0, 1]^2$ ). Then  $f$  not only induces an isomorphism in  $K$ -theory, but also an isomorphism of positive cones, as follows from Proposition 5.1.

iii) Let  $X$  and  $Y$  be the Moore spaces  $M(\mathbf{Z}/3, 2q+11) = S^{2q+11} \cup_3 e^{2q+12}$  and  $M(\mathbf{Z}/3, 2q-1) = S^{2q-1} \cup_3 e^{2q}$  respectively. In [Adams], Adams shows that for  $q$  large enough, there exists a map  $A: X = \Sigma^{12}Y \longrightarrow Y$  such that the induced map  $A^*: \tilde{K}(Y) \longrightarrow \tilde{K}(X)$  is an isomorphism (take  $p = m = 3$ ,  $f = 1$  and  $r = 6$  in Theorem 1.7 and in Lemmas 12.4 and 12.5 of [Adams]). Therefore,  $A$  is a  $K$ -isomorphism between simply connected finite CW-complexes, but it is *not* a homotopy equivalence. The mapping cone  $C_A$  is a non-contractible finite CW-complex with  $\tilde{K}(C_A) = 0$ . (It is non-contractible because its homology is non-trivial.)

iv) In [GrMo], pp. 203-206, a CW-complex  $X = (S^1 \vee S^2) \cup e^3$  is defined, with the property that the inclusion  $i: S^1 = X^{[1]} \hookrightarrow X$  of the 1-skeleton induces an isomorphism in integral homology (and on the level on fundamental groups); however,  $i$  is *not* a homotopy equivalence since  $\pi_2(X) \neq 0$ . Consequently, by the universal coefficient theorem (see Corollary V.7.2 in [Bred]),  $i$  induces an isomorphism in integral cohomology, and, by a direct application of the Atiyah-Hirzebruch spectral sequence, also in  $K$ -theory. In particular,  $i$  is a  $K$ -equivalence, but *not* an equivalence. (As  $C_A$  in the preceding example, the quotient space  $X/X^{[1]}$  has vanishing  $\tilde{K}$ , however it is the closed 3-ball and is therefore contractible.)

Let us finally mention that in [Matt], the positive cone, the  $c$ -cone and the  $\gamma$ -cone are also studied from the rational point of view, and rational  $K$ -theory is considered.

## 6. THE CONES OF THE PRODUCTS $S^n \times S^{2m-1}$

In this section, we will compute the cones for the products  $S^{2n} \times S^{2m-1}$  and  $S^{2n-1} \times S^{2m-1}$ .

We begin with  $S^{2n} \times S^{2m-1}$ . Since  $\tilde{K}(S^{2m-1}) = 0$  and  $K^1(S^{2n}) = 0$ , the answer immediately follows from Proposition 5.5.

**THEOREM 6.1.** *The projection  $p: S^{2n} \times S^{2m-1} \longrightarrow S^{2n}$  induces an isomorphism of positive cones, and, for  $S^{2n} \times S^{2m-1}$ , the  $\gamma$ -cone and the  $c$ -cone coincide with the positive cone:*

$$K_+(S^{2n}) \stackrel{p^*}{\cong} K_+(S^{2n} \times S^{2m-1}) = K_\gamma(S^{2n} \times S^{2m-1}).$$