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Let  $x$  and  $a$  be suitable generators of  $\tilde{K}(S^6)$  and of  $H^6(S^6; \mathbf{Z})$  respectively, and define  $\bar{x} := q^*(x)$  and  $\bar{a} := q^*(a)$ . For obvious dimensional reasons, the Chern classes  $c_1(\bar{x})$  and  $c_2(\bar{x})$  vanish. Moreover, one has  $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$  (see Proposition 2.4), hence  $\text{c-dim}(\bar{x}) = 0$ . On the other hand, we have  $\gamma^1(\bar{x}) = \bar{x} \neq 0$ , so  $\gamma\text{-dim}(\bar{x}) \geq 1$ ; more precisely,  $\gamma^2(\bar{x})$  is  $q^*(-S(3, 2) \cdot x) = q^*(-3x) = \bar{x} \neq 0$  and  $\gamma^3(\bar{x}) = q^*(2S(3, 3) \cdot x) = 0$ , so  $\gamma\text{-dim}(\bar{x}) = 2$ . Consequently,  $M$  is a connected finite CW-complex with a strict inclusion

$$K_\gamma(M) \subsetneq K_c(M).$$

iii) Let  $Z = Y \vee M$  be the wedge of the preceding two examples. It is a 7-dimensional finite connected CW-complex for which none of  $K_\gamma(Z)$  and  $K_c(Z)$  contains the other one. (The product  $Y \times M$  would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

**PROPOSITION 3.5.** *The positive cone, the  $\gamma$ -cone and the  $c$ -cone of a connected finite CW-complex  $X$  are sub-semigroups of  $K(X)$  and homotopy invariants of  $X$ . Moreover, the positive cone is a sub- $\lambda$ -semiring of  $K(X)$ .*

*Proof.* The homotopy invariance is obvious for the three cones. We have already mentioned in the preliminaries that the positive cone is a sub-semiring of  $K(X)$ . It is also clear that it is a sub- $\lambda$ -semiring. The “exponentiality” of  $\gamma_t$  and of  $c$  (the total Chern class) immediately show that the  $\gamma$ -cone and the  $c$ -cone are sub-semigroups of  $K(X)$ .  $\square$

We do not know if in general the  $\gamma$ -cone and the  $c$ -cone are sub- $\lambda$ -semirings of  $K(X)$ .

#### 4. THE POSITIVE CONE OF THE SPHERES

We now intend to compute the positive cone of the spheres. For odd-dimensional spheres, there is nothing to do since  $\tilde{K}(S^{2n+1}) = 0$ . Whereas for even-dimensional spheres, one has  $\tilde{K}(S^{2n}) = \mathbf{Z} \cdot x \cong \mathbf{Z}$ , so we only have to compute  $\text{g-dim}(lx)$  for all integers  $l$ .

By Proposition 2.4, we have

$$c(lx) = c(x)^l = (1 + (-1)^{n-1}(n-1)! \cdot a)^l = 1 + (-1)^{n-1}l(n-1)! \cdot a,$$

where  $a$  is the orientation class of  $S^{2n}$ . Therefore, by Proposition 3.2, we deduce that, for  $l \neq 0$ ,

$$n = \text{c-dim}(lx) \leq \text{g-dim}(lx) \leq \dim(S^{2n})/2 = n,$$

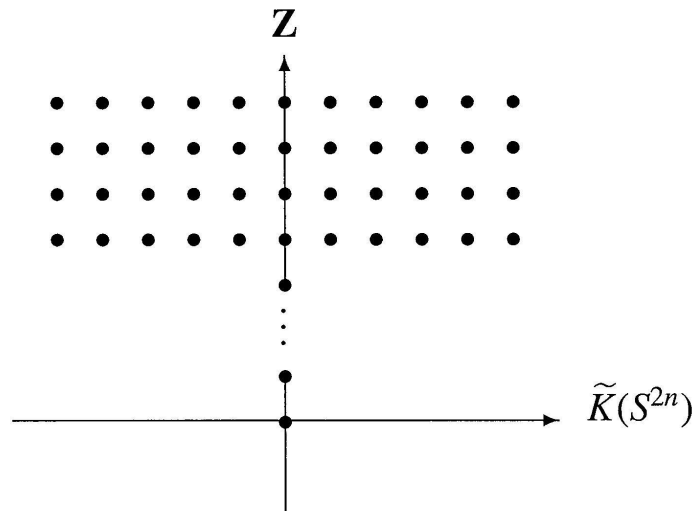
and this shows that  $\text{c-dim}(lx) = \text{g-dim}(lx) = n$ . The sphere  $S^{2n}$  being a torsion-free space, the following theorem follows from Proposition 3.3.

**THEOREM 4.1.** *Let  $x$  be a generator of  $\tilde{K}(S^{2n}) \cong \mathbf{Z}$ . Then, for  $l \in \mathbf{Z}$ ,*

$$\text{g-dim}(lx) = \begin{cases} 0 & \text{if } l = 0 \\ n & \text{otherwise.} \end{cases}$$

Moreover, the positive cone, the  $c$ -cone and the  $\gamma$ -cone of  $S^{2n}$  coincide:

$$K_+(S^{2n}) = K_c(S^{2n}) = K_\gamma(S^{2n}) = \mathbf{N} \times 0 \cup \{(l, x) \mid l \geq n\} \subset \mathbf{Z} \times \tilde{K}(S^{2n}).$$



There is another, purely homotopic, proof of the theorem. It is based on Bott's celebrated results on the homotopy groups of  $BU(n)$  and Serre's computation of the rational homotopy groups of spheres. Let us also present this proof. We have

$$[S^{2n}, BU(k)] = \pi_{2n}(BU(k)) \quad \text{and} \quad \tilde{K}(S^{2n}) = [S^{2n}, BU] = \pi_{2n}(BU).$$

Consider the long exact sequence of the fibration  $BU(k) \xrightarrow{i_k} BU$ :

$$\dots \rightarrow \pi_{2n}(U/U(k)) \rightarrow \pi_{2n}(BU(k)) \xrightarrow{(i_k)_*} \pi_{2n}(BU) \rightarrow \pi_{2n-1}(U/U(k)) \rightarrow \dots$$

The fibre  $U/U(k)$  of  $i_k$  is  $2k$ -connected (see for example [MiTo], p.216) and it follows that  $(i_k)_*$  is an isomorphism for  $n \leq k$ . According to Bott [Bott2], we have  $\pi_{2n}(BU) \cong \mathbf{Z}$ . It is well-known that for  $k < n$ , the group  $\pi_{2n}(BU(k))$  is finite. Let us however give a short proof of this result.

LEMMA 4.2. For  $m \geq 2k + 1$ , the group  $\pi_m(BU(k))$  is finite.

*Proof.* We fix  $m \geq 3$ . The fibration  $BU(k-1) \longrightarrow BU(k)$ , with fibre  $S^{2k-1}$ , yields the following long exact sequence in homotopy:

$$\dots \rightarrow \pi_m(S^{2k-1}) \rightarrow \pi_m(BU(k-1)) \rightarrow \pi_m(BU(k)) \rightarrow \pi_{m-1}(S^{2k-1}) \rightarrow \dots$$

By Serre [Serre],  $\pi_j(S^{2k-1})$  is finite for  $j \neq 2k-1$ , and we can conclude by induction over  $k$  (with  $k \geq 1$  and  $2k+1 \leq m$ ), since when  $k=1$ , one has  $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$  for  $m \geq 3$ .  $\square$

From this, we now infer that the image of  $(i_k)_*$  is zero for  $k < n$ . This implies that  $\text{g-dim}(lx) = n$  when  $l \neq 0$ , and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital  $C^*$ -algebras of type AF by means of their  $K$ -theory, their positive cone and the  $K$ -theory class [1] of the unit (see [Bla1]), it is important to single out the fact that the positive cone of  $S^{2n}$  and that of  $S^{2m}$  are non-isomorphic as monoids if  $n$  is different from  $m$ . (There is no need here to distinguish the  $K$ -theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For  $n \geq 1$ , let  $M_n$  denote the positive cone of  $S^{2n}$  (identified as above with a sub-monoid of  $\mathbf{Z}^2$ , in order to designate its elements). The abelian monoid  $M_n$  has a minimal set  $A_n$  of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0, 1)\} \cup \{(k, n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function  $\sigma: A_n \longrightarrow \{2, 3, \dots\}$  defined, for  $x \in A_n$ , by

$$\sigma(x) := \min \{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}.$$

Clearly, such an  $l$  exists for any  $x \in A_n$  and  $\sigma(A_n) = \{2, 2n\}$ . Since  $A_n$  and  $\sigma$  are isomorphism invariants of  $M_n$ , this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words,  $K(S^{2n-1}) = \mathbf{Z}$  and  $K_+(S^{2n-1}) = \mathbf{N}$ .

## 5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the  $\gamma$ -cone and the  $c$ -cone.

The following result is obvious.