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**SPHERES** 

Autor: MATTHEY, Michel / SUTER, Ulrich

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Let x and a be suitable generators of  $\widetilde{K}(S^6)$  and of  $H^6(S^6; \mathbb{Z})$  respectively, and define  $\bar{x} := q^*(x)$  and  $\bar{a} := q^*(a)$ . For obvious dimensional reasons, the Chern classes  $c_1(\bar{x})$  and  $c_2(\bar{x})$  vanish. Moreover, one has  $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$  (see Proposition 2.4), hence  $\operatorname{c-dim}(\bar{x}) = 0$ . On the other hand, we have  $\gamma^1(\bar{x}) = \bar{x} \neq 0$ , so  $\gamma$ -dim $(\bar{x}) \geq 1$ ; more precisely,  $\gamma^2(\bar{x})$  is  $q^*(-S(3,2)\cdot x) = q^*(-3x) = \bar{x} \neq 0$  and  $\gamma^3(\bar{x}) = q^*(2S(3,3)\cdot x) = 0$ , so  $\gamma$ -dim $(\bar{x}) = 2$ . Consequently, M is a connected finite CW-complex with a strict inclusion

$$K_{\gamma}(M) \subsetneq K_{c}(M)$$
.

iii) Let  $Z = Y \vee M$  be the wedge of the preceding two examples. It is a 7-dimensional finite connected CW-complex for which none of  $K_{\gamma}(Z)$  and  $K_{c}(Z)$  contains the other one. (The product  $Y \times M$  would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

PROPOSITION 3.5. The positive cone, the  $\gamma$ -cone and the c-cone of a connected finite CW-complex X are sub-semigroups of K(X) and homotopy invariants of X. Moreover, the positive cone is a sub- $\lambda$ -semiring of K(X).

*Proof.* The homotopy invariance is obvious for the three cones. We have already mentioned in the preliminaries that the positive cone is a sub-semiring of K(X). It is also clear that it is a sub- $\lambda$ -semiring. The "exponentiality" of  $\gamma_t$  and of c (the total Chern class) immediately show that the  $\gamma$ -cone and the c-cone are sub-semigroups of K(X).

We do not know if in general the  $\gamma$ -cone and the c-cone are sub- $\lambda$ -semirings of K(X).

## 4. The positive cone of the spheres

We now intend to compute the positive cone of the spheres. For odd-dimensional spheres, there is nothing to do since  $\widetilde{K}(S^{2n+1}) = 0$ . Whereas for even-dimensional spheres, one has  $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x \cong \mathbf{Z}$ , so we only have to compute g-dim(lx) for all integers l.

By Proposition 2.4, we have

$$c(lx) = c(x)^{l} = (1 + (-1)^{n-1}(n-1)! \cdot a)^{l} = 1 + (-1)^{n-1}l(n-1)! \cdot a,$$

where a is the orientation class of  $S^{2n}$ . Therefore, by Proposition 3.2, we deduce that, for  $l \neq 0$ ,

$$n = \text{c-dim}(lx) \le \text{g-dim}(lx) \le \text{dim}(S^{2n})/2 = n$$

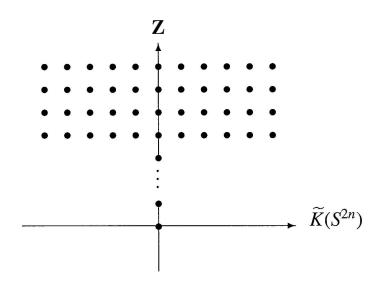
and this shows that  $c-\dim(lx) = g-\dim(lx) = n$ . The sphere  $S^{2n}$  being a torsion-free space, the following theorem follows from Proposition 3.3.

Theorem 4.1. Let x be a generator of  $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$ . Then, for  $l \in \mathbf{Z}$ ,

$$g-\dim(lx) = \begin{cases} 0 & \text{if } l = 0 \\ n & \text{otherwise.} \end{cases}$$

Moreover, the positive cone, the c-cone and the  $\gamma$ -cone of  $S^{2n}$  coincide:

$$K_{+}(S^{2n}) = K_{c}(S^{2n}) = K_{\gamma}(S^{2n}) = \mathbf{N} \times 0 \cup \{(l, x) \mid l \geq n\} \subset \mathbf{Z} \times \widetilde{K}(S^{2n}).$$



There is another, purely homotopic, proof of the theorem. It is based on Bott's celebrated results on the homotopy groups of BU(n) and Serre's computation of the rational homotopy groups of spheres. Let us also present this proof. We have

$$[S^{2n}, BU(k)] = \pi_{2n}(BU(k))$$
 and  $\widetilde{K}(S^{2n}) = [S^{2n}, BU] = \pi_{2n}(BU)$ .

Consider the long exact sequence of the fibration  $BU(k) \xrightarrow{i_k} BU$ :

$$\ldots \to \pi_{2n}(U/U(k)) \to \pi_{2n}(BU(k)) \xrightarrow{(i_k)_*} \pi_{2n}(BU) \to \pi_{2n-1}(U/U(k)) \to \ldots$$

The fibre U/U(k) of  $i_k$  is 2k-connected (see for example [MiTo], p. 216) and it follows that  $(i_k)_*$  is an isomorphism for  $n \le k$ . According to Bott [Bott2], we have  $\pi_{2n}(BU) \cong \mathbb{Z}$ . It is well-known that for k < n, the group  $\pi_{2n}(BU(k))$  is finite. Let us however give a short proof of this result.

LEMMA 4.2. For  $m \ge 2k + 1$ , the group  $\pi_m(BU(k))$  is finite.

*Proof.* We fix  $m \ge 3$ . The fibration  $BU(k-1) \longrightarrow BU(k)$ , with fibre  $S^{2k-1}$ , yields the following long exact sequence in homotopy:

$$\ldots \to \pi_m(S^{2k-1}) \to \pi_m(BU(k-1)) \to \pi_m(BU(k)) \to \pi_{m-1}(S^{2k-1}) \to \ldots$$

By Serre [Serre],  $\pi_j(S^{2k-1})$  is finite for  $j \neq 2k-1$ , and we can conclude by induction over k (with  $k \geq 1$  and  $2k+1 \leq m$ ), since when k=1, one has  $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$  for  $m \geq 3$ .

From this, we now infer that the image of  $(i_k)_*$  is zero for k < n. This implies that  $g\text{-}\dim(lx) = n$  when  $l \neq 0$ , and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital  $C^*$ -algebras of type AF by means of their K-theory, their positive cone and the K-theory class [1] of the unit (see [Bla1]), it is important to single out the fact that the positive cone of  $S^{2n}$  and that of  $S^{2m}$  are non-isomorphic as monoids if n is different from m. (There is no need here to distinguish the K-theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For  $n \ge 1$ , let  $M_n$  denote the positive cone of  $S^{2n}$  (identified as above with a sub-monoid of  $\mathbb{Z}^2$ , in order to designate its elements). The abelian monoid  $M_n$  has a minimal set  $A_n$  of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0, 1)\} \cup \{(k, n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function  $\sigma: A_n \longrightarrow \{2, 3, ...\}$  defined, for  $x \in A_n$ , by  $\sigma(x) := \min\{l \ge 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}$ .

Clearly, such an l exists for any  $x \in A_n$  and  $\sigma(A_n) = \{2, 2n\}$ . Since  $A_n$  and  $\sigma$  are isomorphism invariants of  $M_n$ , this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words,  $K(S^{2n-1}) = \mathbb{Z}$  and  $K_+(S^{2n-1}) = \mathbb{N}$ .

# 5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the  $\gamma$ -cone and the c-cone.

The following result is obvious.