4. The positive cone of the spheres

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Let x and a be suitable generators of $\widetilde{K}(S^6)$ and of $H^6(S^6; \mathbf{Z})$ respectively, and define $\bar{x} := q^*(x)$ and $\bar{a} := q^*(a)$. For obvious dimensional reasons, the Chern classes $c_1(\bar{x})$ and $c_2(\bar{x})$ vanish. Moreover, one has $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$ (see Proposition 2.4), hence c-dim$(\bar{x}) = 0$. On the other hand, we have $\gamma^1(\bar{x}) = \bar{x} \neq 0$, so γ-dim$(\bar{x}) \geq 1$; more precisely, $\gamma^2(\bar{x})$ is $q^*(-S(3,2) \cdot x) = q^*(-3x) = \bar{x} \neq 0$ and $\gamma^3(\bar{x}) = q^*(2S(3,3) \cdot x) = 0$, so γ-dim$(\bar{x}) = 2$. Consequently, M is a connected finite CW-complex with a strict inclusion

$$K_\gamma(M) \subsetneq K_c(M).$$

iii) Let $Z = Y \vee M$ be the wedge of the preceding two examples. It is a 7-dimensional finite connected CW-complex for which none of $K_\gamma(Z)$ and $K_c(Z)$ contains the other one. (The product $Y \times M$ would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

PROPOSITION 3.5. *The positive cone, the γ-cone and the c-cone of a connected finite CW-complex X are sub-semigroups of $K(X)$ and homotopy invariants of X. Moreover, the positive cone is a sub-λ-semiring of $K(X)$.*

Proof. The homotopy invariance is obvious for the three cones. We have already mentioned in the preliminaries that the positive cone is a sub-semiring of $K(X)$. It is also clear that it is a sub-λ-semiring. The "exponentiality" of γ_t and of c (the total Chern class) immediately show that the γ-cone and the c-cone are sub-semigroups of $K(X)$. □

We do not know if in general the γ-cone and the c-cone are sub-λ-semirings of $K(X)$.

4. THE POSITIVE CONE OF THE SPHERES

We now intend to compute the positive cone of the spheres. For odd-dimensional spheres, there is nothing to do since $\widetilde{K}(S^{2n+1}) = 0$. Whereas for even-dimensional spheres, one has $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x \cong \mathbf{Z}$, so we only have to compute g-dim(lx) for all integers l.

By Proposition 2.4, we have

$$c(lx) = c(x)^l = (1 + (-1)^{n-1}(n-1)! \cdot a)^l = 1 + (-1)^{n-1} l(n-1)! \cdot a,$$

where a is the orientation class of S^{2n}. Therefore, by Proposition 3.2, we deduce that, for $l \neq 0$,

$$n = \text{c-dim}(lx) \leq \text{g-dim}(lx) \leq \dim(S^{2n})/2 = n,$$

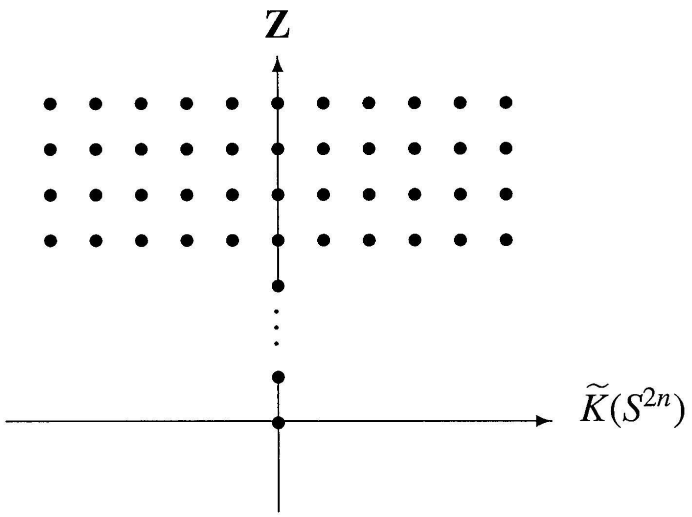
and this shows that $\text{c-dim}(lx) = \text{g-dim}(lx) = n$. The sphere S^{2n} being a torsion-free space, the following theorem follows from Proposition 3.3.

THEOREM 4.1. *Let x be a generator of $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$. Then, for $l \in \mathbf{Z}$,*

$$\text{g-dim}(lx) = \begin{cases} 0 & \textit{if } l = 0 \\ n & \textit{otherwise.} \end{cases}$$

Moreover, the positive cone, the c-cone and the γ-cone of S^{2n} coincide:

$$K_+(S^{2n}) = K_c(S^{2n}) = K_\gamma(S^{2n}) = \mathbf{N} \times 0 \cup \{(l, x) \mid l \geq n\} \subset \mathbf{Z} \times \widetilde{K}(S^{2n}).$$



There is another, purely homotopic, proof of the theorem. It is based on Bott's celebrated results on the homotopy groups of $BU(n)$ and Serre's computation of the rational homotopy groups of spheres. Let us also present this proof. We have

$$[S^{2n}, BU(k)] = \pi_{2n}(BU(k)) \quad \text{and} \quad \widetilde{K}(S^{2n}) = [S^{2n}, BU] = \pi_{2n}(BU).$$

Consider the long exact sequence of the fibration $BU(k) \xrightarrow{i_k} BU$:

$$\ldots \to \pi_{2n}(U/U(k)) \to \pi_{2n}(BU(k)) \xrightarrow{(i_k)_*} \pi_{2n}(BU) \to \pi_{2n-1}(U/U(k)) \to \ldots$$

The fibre $U/U(k)$ of i_k is $2k$-connected (see for example [MiTo], p. 216) and it follows that $(i_k)_*$ is an isomorphism for $n \leq k$. According to Bott [Bott2], we have $\pi_{2n}(BU) \cong \mathbf{Z}$. It is well-known that for $k < n$, the group $\pi_{2n}(BU(k))$ is finite. Let us however give a short proof of this result.

LEMMA 4.2. *For $m \geq 2k+1$, the group $\pi_m(BU(k))$ is finite.*

Proof. We fix $m \geq 3$. The fibration $BU(k-1) \longrightarrow BU(k)$, with fibre S^{2k-1}, yields the following long exact sequence in homotopy:

$$\ldots \to \pi_m(S^{2k-1}) \to \pi_m(BU(k-1)) \to \pi_m(BU(k)) \to \pi_{m-1}(S^{2k-1}) \to \ldots$$

By Serre [Serre], $\pi_j(S^{2k-1})$ is finite for $j \neq 2k-1$, and we can conclude by induction over k (with $k \geq 1$ and $2k+1 \leq m$), since when $k=1$, one has $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$ for $m \geq 3$. □

From this, we now infer that the image of $(i_k)_*$ is zero for $k < n$. This implies that g-dim$(lx) = n$ when $l \neq 0$, and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital C^*-algebras of type AF by means of their K-theory, their positive cone and the K-theory class [1] of the unit (see [Bla1]), it is important to single out the fact that the positive cone of S^{2n} and that of S^{2m} are non-isomorphic as monoids if n is different from m. (There is no need here to distinguish the K-theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For $n \geq 1$, let M_n denote the positive cone of S^{2n} (identified as above with a sub-monoid of $\mathbf{Z}^2$, in order to designate its elements). The abelian monoid M_n has a minimal set A_n of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0,1)\} \cup \{(k,n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function $\sigma\colon A_n \longrightarrow \{2, 3, \ldots\}$ defined, for $x \in A_n$, by

$$\sigma(x) := \min \{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}.$$

Clearly, such an l exists for any $x \in A_n$ and $\sigma(A_n) = \{2, 2n\}$. Since A_n and σ are isomorphism invariants of M_n, this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words, $K(S^{2n-1}) = \mathbf{Z}$ and $K_+(S^{2n-1}) = \mathbf{N}$.

5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the γ-cone and the c-cone.

The following result is obvious.