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THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES

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ABSTRACT. Motivated by Elliott's K-theoretic classification of C^*-algebras of type AF, we compute the positive cone of the K-theory of some spaces. These include the spheres, the products of an odd-dimensional sphere by a sphere, the products of the 2-sphere by a sphere, and of the products $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This amounts to computing the geometric dimension of stable classes of complex vector bundles over these spaces. We establish a few general properties of the positive cone and of approximations to it, the γ-cone and the c-cone. We also get information on the Whitehead product structure in the homotopy groups of $BU(n)$. Moreover, we prove a "doubling formula" for Stirling numbers of the second kind.

1. INTRODUCTION

Let $\mathcal{G}(S)$ be the Grothendieck group completion of an abelian semigroup S, and let $\theta \colon S \longrightarrow \mathcal{G}(S)$ be the corresponding universal homomorphism. The image of θ, denoted by $\mathcal{G}_+(S)$, is a sub-semigroup of $\mathcal{G}(S)$. If S has a zero, in other words if it is an abelian monoid, then $\mathcal{G}_+(S)$ induces a translation invariant preordering on $\mathcal{G}(S)$ (i.e. a reflexive and transitive relation, but not necessarily antisymmetric). The elements of $\mathcal{G}_+(S)$ are called *positive* and $\mathcal{G}_+(S)$ is called the *positive cone* (see [Ell] and [Bla1]). The pair $(\mathcal{G}(S), \mathcal{G}_+(S))$ is an isomorphism invariant of S, and a basic question is: to what extent does this invariant characterize the abelian semigroup S ?

The above notions are of interest in connection with the classification problem of C^*-algebras. For a unital C^*-algebra A, let $S = \mathrm{Proj}(A)$ be the abelian monoid of equivalence classes of projectors in the matrix algebra $\mathbf{M}_\infty(A)$. The K-theory of A, denoted by $K_0(A)$ or $K(A)$, is by definition

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