Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SYMPLECTIC CHARACTERISTIC CLASSES
Autor: BUSCH, Cornelia
Kapitel: 2.2 Symplectic characteristic classes and Chern classes
DOI: https://doi.org/10.5169/seals-65431

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 12.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$\phi(\widetilde{\rho}(z))$ of any generator $z \in \mathbf{Z}/p\mathbf{Z}$ is conjugate to a $Y \in \mathrm{Sp}(2n, \mathbf{Z})$. Then $d_j(\rho) = \widetilde{\rho}^*(c_j) = c_j(\rho)$. We define the total Chern class of a representation $\widetilde{\rho}$ to be

$$c(\widetilde{\rho}) := 1 + c_1(\widetilde{\rho}) + c_2(\widetilde{\rho}) + \cdots + c_n(\widetilde{\rho}).$$

It has the well-known properties $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$, $c(m\rho) = c(\rho)^m$, where ρ, σ are representations and m is a positive integer.

2.2 Symplectic characteristic classes and Chern classes

THEOREM 2.1. *Let p be an odd prime. Then for any $n = 1, \ldots, (p-1)/2$ there exists a representation $\widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ such that the n-th Chern class $c_n(\widetilde{\rho})$ is nonzero and the representation $\phi \circ \widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$.*

The representation $\widetilde{\rho}$ factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ if the image $\widetilde{\rho}(z)$ of a generator $z \in \mathbf{Z}/p\mathbf{Z}$ satisfies the condition stated in Theorem 1.2. Then, because $c_n(\widetilde{\rho}) \neq 0$, we have $d_n(\rho) \neq 0$ where $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$ is the representation corresponding to $\widetilde{\rho}$.

Proof of Theorem 2.1. Let $\mathcal{U}$ be the set of subsets $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$ of cardinality $|\mathcal{I}| = (p-1)/2$, and $j \in \mathcal{I}$ implies $p - j \notin \mathcal{I}$. The cardinality of $\mathcal{U}$ is $2^{(p-1)/2}$. We always assume the elements $j \in \mathcal{I}$ to be represented by integers j with $1 \leqslant j < p$. Note that we will use the same notation for the elements of $\mathcal{I}$ and their representatives. For $j = 1, \ldots, p-1$ let $\widetilde{\rho}_j\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}(1)$ be the one-dimensional representation with $\widetilde{\rho}_j(z) := e^{j2\pi i/p}$ for a fixed generator $z \in \mathbf{Z}/p\mathbf{Z}$. For a given $\mathcal{I}$ we define $\widetilde{\rho}_{\mathcal{I}}$ to be the direct sum of the representations $\widetilde{\rho}_j$, $j \in \mathcal{I}$. Let $x := c_1(\widetilde{\rho}_1)$, then the total Chern class of $\widetilde{\rho}_{\mathcal{I}}$ is

$$c(\widetilde{\rho}_{\mathcal{I}}) = c\Big(\bigoplus_{j \in \mathcal{I}} \widetilde{\rho}_j\Big) = \prod_{j \in \mathcal{I}} (1 + jx).$$

The representations $\widetilde{\rho}_{\mathcal{I}}$ are those which factor through $\mathrm{Sp}(p-1, \mathbf{Z})$. For a given $\mathcal{I} \in \mathcal{U}$ we define $-\mathcal{I} := \{p - j \mid j \in \mathcal{I}\}$. Then $-\mathcal{I} \in \mathcal{U}$ and $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we get $c(\widetilde{\rho}_{\mathcal{I}})c(\widetilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$. The n-th Chern class $c_n(\widetilde{\rho}_{\mathcal{I}})$ is nonzero if and only if the coefficient a_n of x^n in the total Chern class $c(\widetilde{\rho}_{\mathcal{I}})$ is nonzero. Let $\mathcal{I} := \{j_1, \ldots, j_{(p-1)/2}\} \in \mathcal{U}$; then we define

$$\mathcal{I}_l := \{j_1, \ldots, j_{l-1}, -j_l, j_{l+1}, \ldots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that $1 \leqslant n \leqslant (p-1)/2$ exists such that for each set $\mathcal{I} \in \mathcal{U}$ the coefficient a_n of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ is zero. It is impossible that $n = (p-1)/2$

because $a_{(p-1)/2}$ is the product of the $j \in \mathcal{I}$ and therefore nonzero. Now let $n \neq 0$, $n \neq (p-1)/2$; then we define for any $l = 1, \ldots, (p-1)/2$

$$b_n^l := \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J| = n}} \prod_{j \in J} j, \qquad b_0^l := 1 .$$

Then the coefficient of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ is $a_n = b_n^l + j_l b_{n-1}^l$. Because of our assumption, the coefficients of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ and in $c(\widetilde{\rho}_{\mathcal{I}_l})$ are $b_n^l + j_l b_{n-1}^l = 0$ and $b_n^l - j_l b_{n-1}^l = 0$ respectively. This implies that $b_n^l = 0$, $b_{n-1}^l = 0$ and

$$\begin{aligned} a_{n+1} &= \sum_{\substack{J \subseteq \mathcal{I} \\ |J| = n+1}} \prod_{j \in J} j \\ &= \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} \Big(j_l \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J| = n}} \prod_{j \in J} j \Big) = \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} j_l \, b_n^l = 0 . \end{aligned}$$

The factor $1/(n+1)$ appears because in the second line we have $n+1$ times each term appearing in the sum of the first line. Therefore $a_{n+1} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, and by induction we get $a_{(p-1)/2} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, which is impossible. $\square$

Let $\mathrm{Sp}(\mathbf{Z}) := \bigcup_{n \geqslant 1} \mathrm{Sp}(2n, \mathbf{Z})$.

THEOREM 2.2. *For every $j \geqslant 1$, $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ has infinite order.*

Proof. This theorem is a corollary of Theorem 2.1. A consequence of the stability result stated in section 2.1 is that for $p - 1 > 8j + 8$ the inclusion

$$\mathrm{Sp}(p-1, \mathbf{Z}) \longrightarrow \mathrm{Sp}(\mathbf{Z})$$

induces an isomorphism

$$\mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} \mathrm{H}^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}) .$$

In Theorem 2.1 we have shown that for any odd prime p and any integer $j = 1, \ldots, (p-1)/2$ a representation $\widetilde{\rho}_{\mathcal{I}} \colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ exists that factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ and for which the j-th Chern class $c_j(\widetilde{\rho}_{\mathcal{I}})$ is nonzero. Then the j-th symplectic class $d_j(\rho_{\mathcal{I}})$ is also nonzero. Here the

representation $\rho_{\mathcal{I}}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}\,(p-1,\mathbf{Z})$ is the one corresponding to $\widetilde{\rho}_{\mathcal{I}}$. We have an induced homomorphism

$$\begin{aligned} \rho_{\mathcal{I}}^*\colon \mathrm{H}^{2j}(\mathrm{Sp}\,(p-1,\mathbf{Z}),\mathbf{Z}) &\longrightarrow \mathrm{H}^{2j}(\mathbf{Z}/p\mathbf{Z},\mathbf{Z}) \\ d_j(\mathbf{Z}) &\longmapsto d_j(\rho_{\mathcal{I}})\,. \end{aligned}$$

Herewith for any p the class $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}\,(p-1,\mathbf{Z}),\mathbf{Z})$ is nonzero and has either infinite order or finite order divisible by p, since it restricts nontrivially to $\mathrm{H}^{2j}(\mathbf{Z}/p\mathbf{Z},\mathbf{Z})$. This shows that $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}\,(\mathbf{Z}),\mathbf{Z})$ has infinite order. $\square$

This is a new proof of a result of A. Borel [3]. He proved that $\mathrm{H}^*(\mathrm{Sp}\,(\mathbf{Z}),\mathbf{Q}) = \mathbf{Q}[d_1,d_3,\dots]$. Moreover, each d_{2i} can be expressed as a polynomial in the d_{2j+1}'s. This implies that all the $d_i(\mathbf{Z})$'s have infinite order.

REFERENCES

[1] ALEXANDER, J. P., P. E. CONNER and G. C. HAMRICK. *Odd order group actions and Witt classifications of innerproducts.* Lecture Notes in Mathematics *625.* Springer, 1977.

[2] ALEXANDER, J. P., P. E. CONNER, G. C. HAMRICK and J. W. VICK. Witt classes of integral representations of an abelian p-group. *Bull. Amer. Math. Soc. 80* (1974), 1179–1182.

[3] BOREL, A. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4) 7* (1974), 235–272.

[4] BROWN, K. S. Euler characteristics of discrete groups and G-spaces. *Invent. Math. 27* (1974), 229–264.

[5] BÜRGISSER, B. Elements of finite order in symplectic groups. *Arch. Math. (Basel) 39* (1982), 501–509.

[6] BUSCH, C. Symplectic characteristic classes and the Farrell cohomology of $\mathrm{Sp}\,(p-1,\mathbf{Z})$. Dissertation ETH no. 13506, ETH Zurich (2000).

[7] CHARNEY, R. A generalization of a theorem of Vogtmann. *J. Pure Appl. Algebra 44* (1987), 107–125.

[8] GARBANATI, D. A. Unit signatures, and even class numbers, and relative class numbers. *J. reine angew. Math. 274–275* (1975), 376–384.

[9] NEUKIRCH, J. *Algebraische Zahlentheorie.* Springer, 1992.

[10] —— *Class Field Theory.* Grundlehren der mathematischen Wissenschaften 280. Springer, 1986.

[11] SJERVE, D. and Q. YANG. Conjugacy classes of p-torsion in $\mathrm{Sp}_{p-1}(\mathbf{Z})$. *J. Algebra 195* (1997), 580–603.