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$\phi(\tilde{\rho}(z))$ of any generator $z \in \mathbf{Z}/p\mathbf{Z}$ is conjugate to a $Y \in \text{Sp}(2n, \mathbf{Z})$. Then $d_j(\rho) = \tilde{\rho}^*(c_j) = c_j(\rho)$. We define the total Chern class of a representation $\tilde{\rho}$ to be

$$c(\tilde{\rho}) := 1 + c_1(\tilde{\rho}) + c_2(\tilde{\rho}) + \cdots + c_n(\tilde{\rho}).$$

It has the well-known properties $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$, $c(m\rho) = c(\rho)^m$, where ρ, σ are representations and m is a positive integer.

2.2 SYMPLECTIC CHARACTERISTIC CLASSES AND CHERN CLASSES

THEOREM 2.1. *Let p be an odd prime. Then for any $n = 1, \dots, (p-1)/2$ there exists a representation $\tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{U}((p-1)/2)$ such that the n -th Chern class $c_n(\tilde{\rho})$ is nonzero and the representation $\phi \circ \tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{Z})$.*

The representation $\tilde{\rho}$ factors through $\text{Sp}(p-1, \mathbf{Z})$ if the image $\tilde{\rho}(z)$ of a generator $z \in \mathbf{Z}/p\mathbf{Z}$ satisfies the condition stated in Theorem 1.2. Then, because $c_n(\tilde{\rho}) \neq 0$, we have $d_n(\rho) \neq 0$ where $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{Z})$ is the representation corresponding to $\tilde{\rho}$.

Proof of Theorem 2.1. Let \mathcal{U} be the set of subsets $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$ of cardinality $|\mathcal{I}| = (p-1)/2$, and $j \in \mathcal{I}$ implies $p-j \notin \mathcal{I}$. The cardinality of \mathcal{U} is $2^{(p-1)/2}$. We always assume the elements $j \in \mathcal{I}$ to be represented by integers j with $1 \leq j < p$. Note that we will use the same notation for the elements of \mathcal{I} and their representatives. For $j = 1, \dots, p-1$ let $\tilde{\rho}_j: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{U}(1)$ be the one-dimensional representation with $\tilde{\rho}_j(z) := e^{j2\pi i/p}$ for a fixed generator $z \in \mathbf{Z}/p\mathbf{Z}$. For a given \mathcal{I} we define $\tilde{\rho}_{\mathcal{I}}$ to be the direct sum of the representations $\tilde{\rho}_j$, $j \in \mathcal{I}$. Let $x := c_1(\tilde{\rho}_1)$, then the total Chern class of $\tilde{\rho}_{\mathcal{I}}$ is

$$c(\tilde{\rho}_{\mathcal{I}}) = c\left(\bigoplus_{j \in \mathcal{I}} \tilde{\rho}_j\right) = \prod_{j \in \mathcal{I}} (1 + jx).$$

The representations $\tilde{\rho}_{\mathcal{I}}$ are those which factor through $\text{Sp}(p-1, \mathbf{Z})$. For a given $\mathcal{I} \in \mathcal{U}$ we define $-\mathcal{I} := \{p-j \mid j \in \mathcal{I}\}$. Then $-\mathcal{I} \in \mathcal{U}$ and $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we get $c(\tilde{\rho}_{\mathcal{I}})c(\tilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$. The n -th Chern class $c_n(\tilde{\rho}_{\mathcal{I}})$ is nonzero if and only if the coefficient a_n of x^n in the total Chern class $c(\tilde{\rho}_{\mathcal{I}})$ is nonzero. Let $\mathcal{I} := \{j_1, \dots, j_{(p-1)/2}\} \in \mathcal{U}$; then we define

$$\mathcal{I}_l := \{j_1, \dots, j_{l-1}, -j_l, j_{l+1}, \dots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that $1 \leq n \leq (p-1)/2$ exists such that for each set $\mathcal{I} \in \mathcal{U}$ the coefficient a_n of x^n in $c(\tilde{\rho}_{\mathcal{I}})$ is zero. It is impossible that $n = (p-1)/2$

because $a_{(p-1)/2}$ is the product of the $j \in \mathcal{I}$ and therefore nonzero. Now let $n \neq 0$, $n \neq (p-1)/2$; then we define for any $l = 1, \dots, (p-1)/2$

$$b_n^l := \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j, \quad b_0^l := 1.$$

Then the coefficient of x^n in $c(\tilde{\rho}_{\mathcal{I}})$ is $a_n = b_n^l + j_l b_{n-1}^l$. Because of our assumption, the coefficients of x^n in $c(\tilde{\rho}_{\mathcal{I}})$ and in $c(\tilde{\rho}_{\mathcal{I}_l})$ are $b_n^l + j_l b_{n-1}^l = 0$ and $b_n^l - j_l b_{n-1}^l = 0$ respectively. This implies that $b_n^l = 0$, $b_{n-1}^l = 0$ and

$$\begin{aligned} a_{n+1} &= \sum_{\substack{J \subseteq \mathcal{I} \\ |J|=n+1}} \prod_{j \in J} j \\ &= \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} \left(j_l \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j \right) = \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} j_l b_n^l = 0. \end{aligned}$$

The factor $1/(n+1)$ appears because in the second line we have $n+1$ times each term appearing in the sum of the first line. Therefore $a_{n+1} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, and by induction we get $a_{(p-1)/2} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, which is impossible. \square

Let $\mathrm{Sp}(\mathbf{Z}) := \bigcup_{n \geq 1} \mathrm{Sp}(2n, \mathbf{Z})$.

THEOREM 2.2. *For every $j \geq 1$, $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ has infinite order.*

Proof. This theorem is a corollary of Theorem 2.1. A consequence of the stability result stated in section 2.1 is that for $p-1 > 8j+8$ the inclusion

$$\mathrm{Sp}(p-1, \mathbf{Z}) \longrightarrow \mathrm{Sp}(\mathbf{Z})$$

induces an isomorphism

$$H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}).$$

In Theorem 2.1 we have shown that for any odd prime p and any integer $j = 1, \dots, (p-1)/2$ a representation $\tilde{\rho}_{\mathcal{I}}: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}((p-1)/2)$ exists that factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ and for which the j -th Chern class $c_j(\tilde{\rho}_{\mathcal{I}})$ is nonzero. Then the j -th symplectic class $d_j(\rho_{\mathcal{I}})$ is also nonzero. Here the

representation $\rho_{\mathcal{I}}: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$ is the one corresponding to $\tilde{\rho}_{\mathcal{I}}$. We have an induced homomorphism

$$\begin{aligned} \rho_{\mathcal{I}}^*: H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}) &\longrightarrow H^{2j}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}) \\ d_j(\mathbf{Z}) &\longmapsto d_j(\rho_{\mathcal{I}}). \end{aligned}$$

Herewith for any p the class $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z})$ is nonzero and has either infinite order or finite order divisible by p , since it restricts non-trivially to $H^{2j}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z})$. This shows that $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ has infinite order. \square

This is a new proof of a result of A. Borel [3]. He proved that $H^*(\mathrm{Sp}(\mathbf{Z}), \mathbf{Q}) = \mathbf{Q}[d_1, d_3, \dots]$. Moreover, each d_{2i} can be expressed as a polynomial in the d_{2j+1} 's. This implies that all the $d_i(\mathbf{Z})$'s have infinite order.

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