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2. SYMPLECTIC CHARACTERISTIC CLASSES

2.1 CHARACTERISTIC CLASSES AND REPRESENTATIONS

The previously defined homomorphism $\phi: U(n) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ induces

$$H^*(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H^*(B U(n), \mathbf{Z})$$

such that for $j = 1, \dots, n$ the symplectic class $d_j \in H^{2j}(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z})$ maps (per definition) to the universal Chern class $c_j \in H^{2j}(B U(n), \mathbf{Z})$. It is well-known that

$$H^*(B U(n), \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n],$$

$$H^*(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z}) = \mathbf{Z}[d_1, \dots, d_n].$$

The class $d_j = d_j(\mathbf{R})$ restricts to $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$ for $j = 1, \dots, n$. Note that, strictly speaking, the class $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$ depends also on n . But Charney has proven in [7] that for $n > 2j + 4$ the inclusion

$$\mathrm{Sp}(2n, \mathbf{Z}) \longrightarrow \mathrm{Sp}(2n + 2, \mathbf{Z})$$

induces an isomorphism

$$H_j(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H_j(\mathrm{Sp}(2n + 2, \mathbf{Z}), \mathbf{Z}).$$

It is a consequence of the universal coefficient theorem that her result implies the existence of an isomorphism

$$H^j(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H^j(\mathrm{Sp}(2n + 2, \mathbf{Z}), \mathbf{Z})$$

for $n > 2j + 4$. This implies that $H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$ is independent of n for $n > 4j + 4$. Representations

$$\rho: \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathrm{Sp}(2n, \mathbf{Z}),$$

$$\tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \longrightarrow U(n)$$

induce homomorphisms

$$\rho^*: H^*(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \longrightarrow H^*(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}),$$

$$\tilde{\rho}^*: H^*(B U(n), \mathbf{Z}) \longrightarrow H^*(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}).$$

We define $d_j(\rho) := \rho^* d_j(\mathbf{Z})$, the symplectic class of the representation ρ , and $c_j(\tilde{\rho}) := \tilde{\rho}^*(c_j)$, the Chern class of the representation $\tilde{\rho}$. We can consider any representation $\tilde{\rho}$ of $\mathbf{Z}/p\mathbf{Z}$ in $U(n)$ as a representation $\phi \circ \tilde{\rho}$ of $\mathbf{Z}/p\mathbf{Z}$ in $\mathrm{Sp}(2n, \mathbf{R})$. We say that $\tilde{\rho}$ factors through $\mathrm{Sp}(2n, \mathbf{Z})$ if the image

$\phi(\tilde{\rho}(z))$ of any generator $z \in \mathbf{Z}/p\mathbf{Z}$ is conjugate to a $Y \in \text{Sp}(2n, \mathbf{Z})$. Then $d_j(\rho) = \tilde{\rho}^*(c_j) = c_j(\rho)$. We define the total Chern class of a representation $\tilde{\rho}$ to be

$$c(\tilde{\rho}) := 1 + c_1(\tilde{\rho}) + c_2(\tilde{\rho}) + \cdots + c_n(\tilde{\rho}).$$

It has the well-known properties $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$, $c(m\rho) = c(\rho)^m$, where ρ, σ are representations and m is a positive integer.

2.2 SYMPLECTIC CHARACTERISTIC CLASSES AND CHERN CLASSES

THEOREM 2.1. *Let p be an odd prime. Then for any $n = 1, \dots, (p-1)/2$ there exists a representation $\tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{U}((p-1)/2)$ such that the n -th Chern class $c_n(\tilde{\rho})$ is nonzero and the representation $\phi \circ \tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{Z})$.*

The representation $\tilde{\rho}$ factors through $\text{Sp}(p-1, \mathbf{Z})$ if the image $\tilde{\rho}(z)$ of a generator $z \in \mathbf{Z}/p\mathbf{Z}$ satisfies the condition stated in Theorem 1.2. Then, because $c_n(\tilde{\rho}) \neq 0$, we have $d_n(\rho) \neq 0$ where $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{Sp}(p-1, \mathbf{Z})$ is the representation corresponding to $\tilde{\rho}$.

Proof of Theorem 2.1. Let \mathcal{U} be the set of subsets $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$ of cardinality $|\mathcal{I}| = (p-1)/2$, and $j \in \mathcal{I}$ implies $p-j \notin \mathcal{I}$. The cardinality of \mathcal{U} is $2^{(p-1)/2}$. We always assume the elements $j \in \mathcal{I}$ to be represented by integers j with $1 \leq j < p$. Note that we will use the same notation for the elements of \mathcal{I} and their representatives. For $j = 1, \dots, p-1$ let $\tilde{\rho}_j: \mathbf{Z}/p\mathbf{Z} \rightarrow \text{U}(1)$ be the one-dimensional representation with $\tilde{\rho}_j(z) := e^{j2\pi i/p}$ for a fixed generator $z \in \mathbf{Z}/p\mathbf{Z}$. For a given \mathcal{I} we define $\tilde{\rho}_{\mathcal{I}}$ to be the direct sum of the representations $\tilde{\rho}_j$, $j \in \mathcal{I}$. Let $x := c_1(\tilde{\rho}_1)$, then the total Chern class of $\tilde{\rho}_{\mathcal{I}}$ is

$$c(\tilde{\rho}_{\mathcal{I}}) = c\left(\bigoplus_{j \in \mathcal{I}} \tilde{\rho}_j\right) = \prod_{j \in \mathcal{I}} (1 + jx).$$

The representations $\tilde{\rho}_{\mathcal{I}}$ are those which factor through $\text{Sp}(p-1, \mathbf{Z})$. For a given $\mathcal{I} \in \mathcal{U}$ we define $-\mathcal{I} := \{p-j \mid j \in \mathcal{I}\}$. Then $-\mathcal{I} \in \mathcal{U}$ and $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we get $c(\tilde{\rho}_{\mathcal{I}})c(\tilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$. The n -th Chern class $c_n(\tilde{\rho}_{\mathcal{I}})$ is nonzero if and only if the coefficient a_n of x^n in the total Chern class $c(\tilde{\rho}_{\mathcal{I}})$ is nonzero. Let $\mathcal{I} := \{j_1, \dots, j_{(p-1)/2}\} \in \mathcal{U}$; then we define

$$\mathcal{I}_l := \{j_1, \dots, j_{l-1}, -j_l, j_{l+1}, \dots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that $1 \leq n \leq (p-1)/2$ exists such that for each set $\mathcal{I} \in \mathcal{U}$ the coefficient a_n of x^n in $c(\tilde{\rho}_{\mathcal{I}})$ is zero. It is impossible that $n = (p-1)/2$