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2. Symplectic characteristic classes

2.1 Characteristic classes and representations

The previously defined homomorphism $\phi\colon \mathrm{U}(n) \to \mathrm{Sp}(2n, \mathbf{R})$ induces

$$\mathrm{H}^*(B\,\mathrm{Sp}(2n,\mathbf{R}),\mathbf{Z}) \xrightarrow{\cong} \mathrm{H}^*(B\,\mathrm{U}(n),\mathbf{Z})$$

such that for $j = 1,\dots,n$ the symplectic class $d_j \in \mathrm{H}^{2j}(B\,\mathrm{Sp}(2n,\mathbf{R}),\mathbf{Z})$ maps (per definition) to the universal Chern class $c_j \in \mathrm{H}^{2j}(B\,\mathrm{U}(n),\mathbf{Z})$. It is well-known that

$$\mathrm{H}^*(B\,\mathrm{U}(n),\mathbf{Z}) = \mathbf{Z}[c_1,\dots,c_n]\,,$$

$$\mathrm{H}^*(B\,\mathrm{Sp}(2n,\mathbf{R}),\mathbf{Z}) = \mathbf{Z}[d_1,\dots,d_n]\,.$$

The class $d_j = d_j(\mathbf{R})$ restricts to $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z})$ for $j = 1,\dots,n$. Note that, strictly speaking, the class $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z})$ depends also on n. But Charney has proven in [7] that for $n > 2j+4$ the inclusion

$$\mathrm{Sp}(2n,\mathbf{Z}) \longrightarrow \mathrm{Sp}(2n+2,\mathbf{Z})$$

induces an isomorphism

$$\mathrm{H}_j(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z}) \xrightarrow{\cong} \mathrm{H}_j(\mathrm{Sp}(2n+2,\mathbf{Z}),\mathbf{Z})\,.$$

It is a consequence of the universal coefficient theorem that her result implies the existence of an isomorphism

$$\mathrm{H}^j(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z}) \xrightarrow{\cong} \mathrm{H}^j(\mathrm{Sp}(2n+2,\mathbf{Z}),\mathbf{Z})$$

for $n > 2j+4$. This implies that $\mathrm{H}^{2j}(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z})$ is independent of n for $n > 4j+4$. Representations

$$\rho\colon \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathrm{Sp}(2n,\mathbf{Z})\,,$$

$$\widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathrm{U}(n)$$

induce homomorphisms

$$\rho^*\colon\ \mathrm{H}^*(\mathrm{Sp}(2n,\mathbf{Z}),\mathbf{Z}) \longrightarrow \mathrm{H}^*(\mathbf{Z}/p\mathbf{Z},\mathbf{Z})\,,$$

$$\widetilde{\rho}^*\colon\ \mathrm{H}^*(B\,\mathrm{U}(n),\mathbf{Z}) \longrightarrow \mathrm{H}^*(\mathbf{Z}/p\mathbf{Z},\mathbf{Z})\,.$$

We define $d_j(\rho) := \rho^* d_j(\mathbf{Z})$, the symplectic class of the representation ρ, and $c_j(\widetilde{\rho}) := \widetilde{\rho}^*(c_j)$, the Chern class of the representation $\widetilde{\rho}$. We can consider any representation $\widetilde{\rho}$ of $\mathbf{Z}/p\mathbf{Z}$ in $\mathrm{U}(n)$ as a representation $\phi \circ \widetilde{\rho}$ of $\mathbf{Z}/p\mathbf{Z}$ in $\mathrm{Sp}(2n,\mathbf{R})$. We say that $\widetilde{\rho}$ factors through $\mathrm{Sp}(2n,\mathbf{Z})$ if the image

$\phi(\widetilde{\rho}(z))$ of any generator $z \in \mathbf{Z}/p\mathbf{Z}$ is conjugate to a $Y \in \mathrm{Sp}(2n, \mathbf{Z})$. Then $d_j(\rho) = \widetilde{\rho}^*(c_j) = c_j(\rho)$. We define the total Chern class of a representation $\widetilde{\rho}$ to be

$$c(\widetilde{\rho}) := 1 + c_1(\widetilde{\rho}) + c_2(\widetilde{\rho}) + \cdots + c_n(\widetilde{\rho}).$$

It has the well-known properties $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$, $c(m\rho) = c(\rho)^m$, where ρ, σ are representations and m is a positive integer.

2.2 Symplectic characteristic classes and Chern classes

Theorem 2.1. *Let p be an odd prime. Then for any $n = 1, \dots, (p-1)/2$ there exists a representation $\widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ such that the n-th Chern class $c_n(\widetilde{\rho})$ is nonzero and the representation $\phi \circ \widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$.*

The representation $\widetilde{\rho}$ factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ if the image $\widetilde{\rho}(z)$ of a generator $z \in \mathbf{Z}/p\mathbf{Z}$ satisfies the condition stated in Theorem 1.2. Then, because $c_n(\widetilde{\rho}) \neq 0$, we have $d_n(\rho) \neq 0$ where $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$ is the representation corresponding to $\widetilde{\rho}$.

Proof of Theorem 2.1. Let $\mathcal{U}$ be the set of subsets $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$ of cardinality $|\mathcal{I}| = (p-1)/2$, and $j \in \mathcal{I}$ implies $p - j \notin \mathcal{I}$. The cardinality of $\mathcal{U}$ is $2^{(p-1)/2}$. We always assume the elements $j \in \mathcal{I}$ to be represented by integers j with $1 \leqslant j < p$. Note that we will use the same notation for the elements of $\mathcal{I}$ and their representatives. For $j = 1, \dots, p-1$ let $\widetilde{\rho}_j\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}(1)$ be the one-dimensional representation with $\widetilde{\rho}_j(z) := e^{j2\pi i/p}$ for a fixed generator $z \in \mathbf{Z}/p\mathbf{Z}$. For a given $\mathcal{I}$ we define $\widetilde{\rho}_{\mathcal{I}}$ to be the direct sum of the representations $\widetilde{\rho}_j$, $j \in \mathcal{I}$. Let $x := c_1(\widetilde{\rho}_1)$, then the total Chern class of $\widetilde{\rho}_{\mathcal{I}}$ is

$$c(\widetilde{\rho}_{\mathcal{I}}) = c\Big(\bigoplus_{j\in\mathcal{I}} \widetilde{\rho}_j\Big) = \prod_{j\in\mathcal{I}} (1 + jx).$$

The representations $\widetilde{\rho}_{\mathcal{I}}$ are those which factor through $\mathrm{Sp}(p-1, \mathbf{Z})$. For a given $\mathcal{I} \in \mathcal{U}$ we define $-\mathcal{I} := \{p - j \mid j \in \mathcal{I}\}$. Then $-\mathcal{I} \in \mathcal{U}$ and $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we get $c(\widetilde{\rho}_{\mathcal{I}})c(\widetilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$. The n-th Chern class $c_n(\widetilde{\rho}_{\mathcal{I}})$ is nonzero if and only if the coefficient a_n of x^n in the total Chern class $c(\widetilde{\rho}_{\mathcal{I}})$ is nonzero. Let $\mathcal{I} := \{j_1, \dots, j_{(p-1)/2}\} \in \mathcal{U}$; then we define

$$\mathcal{I}_l := \{j_1, \dots, j_{l-1}, -j_l, j_{l+1}, \dots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that $1 \leqslant n \leqslant (p-1)/2$ exists such that for each set $\mathcal{I} \in \mathcal{U}$ the coefficient a_n of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ is zero. It is impossible that $n = (p-1)/2$