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SYMPLECTIC CHARACTERISTIC CLASSES

by Cornelia BUSCH

ABSTRACT. We present a new proof of the fact that the universal symplectic classes $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ have infinite order. This proof uses only techniques from group cohomology. In order to obtain this result, we determine representations $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}((p-1)/2)$ whose associated representation $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$.

In this article we prerequire some basic notions from the theory of cyclotomic fields. For the reader who is not familiar with this subject we recommend the books of Washington [12] and of Neukirch [9]. An introduction to the arithmetical part is also given in my thesis [6].

This article presents a result of my doctoral thesis, which I wrote at the ETH Zurich under the supervision of G. Mislin, whom I want to thank for his excellent support.

1. THE SYMPLECTIC GROUP

1.1 DEFINITION

Let R be a commutative ring with 1. The general linear group $\mathrm{GL}(n, R)$ is defined to be the multiplicative group of invertible $n \times n$ -matrices over R .

DEFINITION. The *symplectic group* $\mathrm{Sp}(2n, R)$ over the ring R is the subgroup of matrices $Y \in \mathrm{GL}(2n, R)$ that satisfy

$$Y^T J Y = J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ -identity matrix.

It is the group of isometries of the skew-symmetric bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle: R^{2n} \times R^{2n} &\longrightarrow R \\ (x, y) &\longmapsto \langle x, y \rangle := x^T J y. \end{aligned}$$

It follows from a result of Bürgisser [5] that elements of odd prime order p exist in $\text{Sp}(2n, \mathbf{Z})$ if and only if $2n \geq p - 1$.

PROPOSITION 1.1. *The eigenvalues of a matrix $Y \in \text{Sp}(p - 1, \mathbf{Z})$ of odd prime order p are the primitive p -th roots of unity, hence the zeros of the polynomial*

$$m(x) = x^{p-1} + \dots + x + 1.$$

Proof. If λ is an eigenvalue of Y , we have $\lambda = 1$ or $\lambda = \xi$, a primitive p -th root of unity, and the characteristic polynomial of Y divides $x^p - 1$ and has integer coefficients. Since $m(x)$ is irreducible over \mathbf{Q} , the claim follows. \square

1.2 A RELATION BETWEEN $U\left(\frac{p-1}{2}\right)$ AND $\text{Sp}(p - 1, \mathbf{Z})$

Let $X \in U(n)$, i.e., $X \in \text{GL}(n, \mathbf{C})$ and $X^*X = I_n$ where $X^* = \bar{X}^T$ and I_n is the $n \times n$ -identity matrix. We can write $X = A + iB$ with $A, B \in M(n, \mathbf{R})$, the ring of real matrices. We now define the following map

$$\begin{aligned} \phi: U(n) &\longrightarrow \text{Sp}(2n, \mathbf{R}) \\ X = A + iB &\longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} =: \phi(X). \end{aligned}$$

The map ϕ is an injective homomorphism. Moreover, it is well-known that ϕ maps $U(n)$ onto a maximal compact subgroup of $\text{Sp}(2n, \mathbf{R})$. In this section we will prove the following theorem.

THEOREM 1.2. *Let $X \in U((p - 1)/2)$ be of odd prime order p . We define $\phi: U((p - 1)/2) \rightarrow \text{Sp}(p - 1, \mathbf{R})$ as above. Then $\phi(X) \in \text{Sp}(p - 1, \mathbf{R})$ is conjugate to $Y \in \text{Sp}(p - 1, \mathbf{Z})$ if and only if the eigenvalues $\lambda_1, \dots, \lambda_{(p-1)/2}$ of X are such that*

$$\{\lambda_1, \dots, \lambda_{(p-1)/2}, \bar{\lambda}_1, \dots, \bar{\lambda}_{(p-1)/2}\}$$

is a complete set of primitive p -th roots of unity.