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## SYMPLECTIC CHARACTERISTIC CLASSES

by Cornelia BUSCH

**ABSTRACT.** We present a new proof of the fact that the universal symplectic classes  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$  have infinite order. This proof uses only techniques from group cohomology. In order to obtain this result, we determine representations  $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}((p-1)/2)$  whose associated representation  $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{R})$  factors, up to conjugation, through a representation  $\mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$ .

In this article we prerequire some basic notions from the theory of cyclotomic fields. For the reader who is not familiar with this subject we recommend the books of Washington [12] and of Neukirch [9]. An introduction to the arithmetical part is also given in my thesis [6].

This article presents a result of my doctoral thesis, which I wrote at the ETH Zurich under the supervision of G. Mislin, whom I want to thank for his excellent support.

### 1. THE SYMPLECTIC GROUP

#### 1.1 DEFINITION

Let  $R$  be a commutative ring with 1. The general linear group  $\mathrm{GL}(n, R)$  is defined to be the multiplicative group of invertible  $n \times n$ -matrices over  $R$ .

**DEFINITION.** The *symplectic group*  $\mathrm{Sp}(2n, R)$  over the ring  $R$  is the subgroup of matrices  $Y \in \mathrm{GL}(2n, R)$  that satisfy

$$Y^T J Y = J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where  $I_n$  is the  $n \times n$ -identity matrix.

It is the group of isometries of the skew-symmetric bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle: R^{2n} \times R^{2n} &\longrightarrow R \\ (x, y) &\longmapsto \langle x, y \rangle := x^T J y. \end{aligned}$$

It follows from a result of B urgisser [5] that elements of odd prime order  $p$  exist in  $\mathrm{Sp}(2n, \mathbf{Z})$  if and only if  $2n \geq p - 1$ .

**PROPOSITION 1.1.** *The eigenvalues of a matrix  $Y \in \mathrm{Sp}(p - 1, \mathbf{Z})$  of odd prime order  $p$  are the primitive  $p$ -th roots of unity, hence the zeros of the polynomial*

$$m(x) = x^{p-1} + \cdots + x + 1.$$

*Proof.* If  $\lambda$  is an eigenvalue of  $Y$ , we have  $\lambda = 1$  or  $\lambda = \xi$ , a primitive  $p$ -th root of unity, and the characteristic polynomial of  $Y$  divides  $x^p - 1$  and has integer coefficients. Since  $m(x)$  is irreducible over  $\mathbf{Q}$ , the claim follows.  $\square$

## 1.2 A RELATION BETWEEN $U\left(\frac{p-1}{2}\right)$ AND $\mathrm{Sp}(p - 1, \mathbf{Z})$

Let  $X \in U(n)$ , i.e.,  $X \in \mathrm{GL}(n, \mathbf{C})$  and  $X^*X = I_n$  where  $X^* = \bar{X}^T$  and  $I_n$  is the  $n \times n$ -identity matrix. We can write  $X = A + iB$  with  $A, B \in M(n, \mathbf{R})$ , the ring of real matrices. We now define the following map

$$\begin{aligned} \phi: U(n) &\longrightarrow \mathrm{Sp}(2n, \mathbf{R}) \\ X = A + iB &\longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} =: \phi(X). \end{aligned}$$

The map  $\phi$  is an injective homomorphism. Moreover, it is well-known that  $\phi$  maps  $U(n)$  onto a maximal compact subgroup of  $\mathrm{Sp}(2n, \mathbf{R})$ . In this section we will prove the following theorem.

**THEOREM 1.2.** *Let  $X \in U((p - 1)/2)$  be of odd prime order  $p$ . We define  $\phi: U((p - 1)/2) \rightarrow \mathrm{Sp}(p - 1, \mathbf{R})$  as above. Then  $\phi(X) \in \mathrm{Sp}(p - 1, \mathbf{R})$  is conjugate to  $Y \in \mathrm{Sp}(p - 1, \mathbf{Z})$  if and only if the eigenvalues  $\lambda_1, \dots, \lambda_{(p-1)/2}$  of  $X$  are such that*

$$\{\lambda_1, \dots, \lambda_{(p-1)/2}, \bar{\lambda}_1, \dots, \bar{\lambda}_{(p-1)/2}\}$$

*is a complete set of primitive  $p$ -th roots of unity.*

*The condition on the eigenvalues of  $X$  is necessary:* It is an easy computation to show that if  $\lambda_1, \dots, \lambda_{(p-1)/2}$  are the eigenvalues of  $X \in U((p-1)/2)$ , then

$$\lambda_1, \dots, \lambda_{(p-1)/2}, \bar{\lambda}_1, \dots, \bar{\lambda}_{(p-1)/2}$$

are the eigenvalues of  $\phi(X) \in \text{Sp}(p-1, \mathbf{R})$ . So if  $\phi(X) \in \text{Sp}(p-1, \mathbf{R})$  is conjugate to  $Y \in \text{Sp}(p-1, \mathbf{Z})$ , the condition on the eigenvalues of  $X \in U((p-1)/2)$  holds by Proposition 1.1. That the condition on the eigenvalues is also sufficient will be proved in 1.2.2.

Note that  $X_1, X_2 \in U(n)$  are conjugate in  $U(n)$  if and only if  $\phi(X_1), \phi(X_2)$  are conjugate in  $\text{Sp}(2n, \mathbf{R})$ , because  $\phi(U(n))$  is a maximal compact subgroup of  $\text{Sp}(2n, \mathbf{R})$ . The eigenvalues of a unitary matrix  $X$  determine the conjugacy class of  $X$  in  $U((p-1)/2)$ . We will take any  $Y \in \text{Sp}(p-1, \mathbf{Z})$  of prime order  $p$  and show, assuming  $Y$  is conjugate in  $\text{Sp}(p-1, \mathbf{R})$  to  $\phi(X)$ , how to compute the eigenvalues of  $X \in U((p-1)/2)$ . Then we will prove that if we run through the conjugacy classes of matrices  $Y \in \text{Sp}(p-1, \mathbf{Z})$  of prime order  $p$ , we will run through the conjugacy classes of matrices  $X \in U((p-1)/2)$  that satisfy the necessary condition. An interesting corollary is the following (see also 1.2.2).

**COROLLARY 1.3.** *The number of conjugacy classes of elements of order  $p$  in  $\text{Sp}(p-1, \mathbf{Z})$  that are conjugate in  $\text{Sp}(p-1, \mathbf{R})$  to elements of the form  $\phi(X)$ , where  $X \in U((p-1)/2)$ , is greater or equal to  $2^{(p-1)/2}$ .*

### 1.2.1 INVARIANT SUBSPACES

Each matrix  $Y \in \text{Sp}(p-1, \mathbf{Z})$  of odd prime order  $p$  defines an isomorphism  $\sigma: \mathbf{Z}^{p-1} \rightarrow \mathbf{Z}^{p-1}$ , which is an isometry of the skew-symmetric bilinear form  $q: \mathbf{Z}^{p-1} \times \mathbf{Z}^{p-1} \rightarrow \mathbf{Z}$  defined by  $q(x, y) := \langle x, y \rangle = x^T J y$  where  $x, y \in \mathbf{Z}^{p-1}$  and  $J$  is like in the definition of the symplectic group. From now on we will sometimes take the  $\mathbf{R}$ -linear or the  $\mathbf{C}$ -linear extensions of  $\sigma$  and of  $q$  without making any remark. But this will always be clear from the context.

Let  $v_j \in \mathbf{C}^{p-1}$  be an eigenvector corresponding to the eigenvalue  $\xi^j := e^{j2\pi i/p}$  of the  $\mathbf{C}$ -linear extension of  $\sigma$ . Then the complex conjugate  $\bar{v}_j$  is an eigenvector to the eigenvalue  $\xi^{-j}$  because  $\sigma$  is given by a real matrix. The real vectors  $v_j + \bar{v}_j$  and  $-i(v_j - \bar{v}_j)$  span a  $\sigma$ -invariant subspace of  $\mathbf{R}^{p-1}$ , which we will denote by  $V_j$ . The dimension of  $V_j$  is 2 and  $\mathbf{R}^{p-1} = V_1 \oplus \dots \oplus V_{(p-1)/2}$ . The space  $V_j \otimes_{\mathbf{R}} \mathbf{C}$  is the sum of the eigenspaces corresponding to  $\xi^j$  and  $\xi^{-j}$ .



DEFINITION. We define the *sign* of  $V_j$  to be

$$\text{sign}(V_j) := \text{sign } q(x, \sigma(x)),$$

where  $x \in V_j$  is any nonzero element.

LEMMA 1.4. *The sign  $\text{sign}(V_j)$  is well-defined, i.e., independent of the choice of  $x$ .*

*Proof.* Let  $0 \neq x := \alpha(v_j + \bar{v}_j) + \beta(-i(v_j - \bar{v}_j)) \in V_j$  where  $\alpha, \beta \in \mathbf{R}$  and  $v_j, \bar{v}_j$  as above. Then a simple computation shows that

$$q(x, \sigma(x)) = -2i(\alpha^2 + \beta^2)q(v_j, \bar{v}_j) \sin \theta_j \neq 0,$$

with  $\theta_j := j2\pi/p$ . Therefore,  $\text{sign } q(x, \sigma(x))$  does not depend on the choice of  $0 \neq x \in V_j$ .  $\square$

For  $x \in V_j, y \in V_k$  with  $j \neq k, j, k = 1, \dots, (p-1)/2$ , we have  $q(x, y) = 0$ . Therefore  $q$  is nondegenerate on  $V_j$  and  $q(v_j, \bar{v}_j) = -q(\bar{v}_j, v_j) \neq 0$ . Because  $\sin \theta_j > 0$ , we have

$$\text{sign}(V_j) = \text{sign}(-iq(v_j, \bar{v}_j)).$$

This equation implies that  $-i \text{sign}(V_j)q(v_j, \bar{v}_j)$  is positive. We define a new basis of  $V_j$  by:

$$u_j := (-2i \text{sign}(V_j) q(v_j, \bar{v}_j))^{-1/2}(v_j + \bar{v}_j),$$

$$\tilde{u}_j := -\text{sign}(V_j) (-2i \text{sign}(V_j) q(v_j, \bar{v}_j))^{-1/2}(-i(v_j - \bar{v}_j)).$$

LEMMA 1.5. *The vectors  $u_1, \dots, u_{(p-1)/2}, \tilde{u}_1, \dots, \tilde{u}_{(p-1)/2}$  form a symplectic basis of  $\mathbf{R}^{p-1}$ .*

*Proof.* It is clear that this is a basis of  $\mathbf{R}^{p-1}$ . For  $i \neq j$  with  $i, j = 1, \dots, (p-1)/2$

$$q(u_i, u_j) = q(\tilde{u}_i, \tilde{u}_j) = q(u_i, \tilde{u}_j) = 0,$$

$$q(u_j, \tilde{u}_j) = 1.$$

This shows that the basis  $u_1, \dots, u_{(p-1)/2}, \tilde{u}_1, \dots, \tilde{u}_{(p-1)/2}$  is symplectic.  $\square$

The matrix corresponding to  $\sigma|_{V_j}: V_j \rightarrow V_j$  in the basis  $u_j, \tilde{u}_j$  is the following:

$$\begin{pmatrix} \cos \theta_j & -\text{sign}(V_j) \sin \theta_j \\ \text{sign}(V_j) \sin \theta_j & \cos \theta_j \end{pmatrix}.$$

We want to write this matrix in the form

$$\begin{pmatrix} \cos \vartheta_j & \sin \vartheta_j \\ -\sin \vartheta_j & \cos \vartheta_j \end{pmatrix},$$

because in this case  $\sigma: \mathbf{R}^{p-1} \rightarrow \mathbf{R}^{p-1}$  is given in the basis  $u_1, \dots, u_{(p-1)/2}, \tilde{u}_1, \dots, \tilde{u}_{(p-1)/2}$  by the image of a diagonal matrix in  $X \in U((p-1)/2)$  with the  $e^{i\vartheta_j}$ ,  $j = 1, \dots, (p-1)/2$ , being the eigenvalues of  $X$ . Comparing both  $2 \times 2$ -matrices we see that we should put

$$\vartheta_j := \begin{cases} \theta_j & \text{if } \text{sign}(V_j) = -1 \\ 2\pi - \theta_j & \text{if } \text{sign}(V_j) = +1. \end{cases}$$

This proves the following

**PROPOSITION 1.6.** *Let  $Y \in \text{Sp}(p-1, \mathbf{Z})$  of odd prime order  $p$  define an isometry  $\sigma: \mathbf{Z}^{p-1} \rightarrow \mathbf{Z}^{p-1}$ . Let  $\xi := e^{i2\pi/p}$ ,  $\mathbf{R}^{p-1} = V_1 \oplus \dots \oplus V_{(p-1)/2}$  where  $V_j$ ,  $j = 1, \dots, (p-1)/2$ , is the invariant subspace corresponding to the eigenvalues  $\xi^j, \xi^{p-j}$  of the extension of  $\sigma$  to an isomorphism of  $\mathbf{R}^{p-1}$ . Then there exists  $X \in U((p-1)/2)$  such that  $Y$  is conjugate to  $\phi(X) \in \text{Sp}(p-1, \mathbf{R})$ . Moreover,*

*if  $\text{sign}(V_j) = -1$  then  $\xi^j$  is an eigenvalue of  $X$ , and*

*if  $\text{sign}(V_j) = 1$  then  $\xi^{-j}$  is an eigenvalue of  $X$ .*

### 1.2.2 THE PROOF OF THEOREM 1.2

*It remains to show that the condition on the eigenvalues of  $X \in U((p-1)/2)$  is sufficient.* We put  $\mathbf{Z}/2\mathbf{Z} = \{\pm 1\}$ . Let  $\mathcal{M}$  be the set of  $Y \in \text{Sp}(p-1, \mathbf{Z})$  of odd prime order  $p$ . We define a mapping

$$\begin{aligned} \psi: \mathcal{M} &\longrightarrow (\mathbf{Z}/2\mathbf{Z})^{(p-1)/2} \\ Y &\longmapsto (\text{sign}(V_1), \dots, \text{sign}(V_{(p-1)/2})) \end{aligned}$$

where  $V_j$  and  $\text{sign}(V_j)$ ,  $j = 1, \dots, (p-1)/2$ , are defined as above. It follows from Proposition 1.6 that the necessary condition in Theorem 1.2 is sufficient if and only if  $\psi$  is surjective. Therefore we now have to prove the surjectivity of  $\psi$ . First we will prove that in each conjugacy class of matrices of order  $p$  in  $\text{Sp}(p-1, \mathbf{Z}[1/p])$  one can find a matrix in  $\text{Sp}(p-1, \mathbf{Z})$ . Let  $\mathcal{M}_p$  be the set of matrices of order  $p$  in  $\text{Sp}(p-1, \mathbf{Z}[1/p])$ . With the same procedure as for  $Y \in \mathcal{M}$ , we can define  $V_j$ ,  $\text{sign}(V_j)$ ,  $j = 1, \dots, (p-1)/2$ , for  $Y_p \in \mathcal{M}_p$ , and we get statements for  $\text{Sp}(p-1, \mathbf{Z}[1/p])$  that are similar to those for

$\mathrm{Sp}(p-1, \mathbf{Z})$ . We will show the surjectivity of the mapping

$$\begin{aligned} \psi_p: \mathcal{M}_p &\longrightarrow (\mathbf{Z}/2\mathbf{Z})^{(p-1)/2} \\ Y_p &\longmapsto (\mathrm{sign}(V_1), \dots, \mathrm{sign}(V_{(p-1)/2})) . \end{aligned}$$

Then we have shown that  $\psi$  is surjective since matrices of  $\mathcal{M}_p$  that are in the same conjugacy class have the same image under  $\psi_p$ .

Let  $P$  be the set of pairs  $(\mathfrak{a}, a)$ , where  $0 \neq \mathfrak{a} \subseteq \mathbf{Z}[\xi]$  is an ideal and  $a \in \mathbf{Z}[\xi]$  such that  $\mathfrak{a}\bar{a} = (a) \subseteq \mathbf{Z}[\xi]$  is a principal ideal. The bar denotes complex conjugation and  $\bar{a} = \{\bar{\alpha} \mid \alpha \in \mathfrak{a}\}$ . Let  $P_p$  be the set of pairs  $(\mathfrak{a}_p, a)$ , where  $0 \neq \mathfrak{a}_p \subseteq \mathbf{Z}[1/p][\xi]$  is an ideal and  $a \in \mathbf{Z}[1/p][\xi]$  such that  $\mathfrak{a}_p\bar{a}_p = (a) \subseteq \mathbf{Z}[1/p][\xi]$  is a principal ideal. We define an equivalence relation on  $P$  and on  $P_p$ :

$$\begin{aligned} (\mathfrak{a}, a) \sim (\mathfrak{b}, b) &\Leftrightarrow \exists \lambda, \mu \in \mathbf{Z}[\xi] \setminus \{0\} \text{ such that} \\ &\lambda\mathfrak{a} = \mu\mathfrak{b} \text{ and } \lambda\bar{\lambda}a = \mu\bar{\mu}b \end{aligned}$$

$$\begin{aligned} (\mathfrak{a}_p, a) \sim (\mathfrak{b}_p, b) &\Leftrightarrow \exists \lambda, \mu \in \mathbf{Z}[1/p][\xi] \setminus \{0\} \text{ such that} \\ &\lambda\mathfrak{a}_p = \mu\mathfrak{b}_p \text{ and } \lambda\bar{\lambda}a = \mu\bar{\mu}b . \end{aligned}$$

We denote by  $[\mathfrak{a}, a]$  and  $[\mathfrak{a}_p, a]$  the equivalence class of  $(\mathfrak{a}, a)$  and  $(\mathfrak{a}_p, a)$  respectively. Moreover,  $\mathcal{P}$  and  $\mathcal{P}_p$  denote the sets of equivalence classes in  $P$  and  $P_p$  respectively. The sets of equivalence classes  $\mathcal{P}$  and  $\mathcal{P}_p$  are abelian groups. The multiplication is given by  $[\mathfrak{a}, a][\mathfrak{b}, b] = [\mathfrak{a}\mathfrak{b}, ab]$ , the units in  $\mathcal{P}$  and  $\mathcal{P}_p$  are  $[\mathbf{Z}[\xi], 1]$  and  $[\mathbf{Z}[1/p][\xi], 1]$  respectively, and the inverse of  $[\mathfrak{a}, a]$  is  $[\bar{\mathfrak{a}}, a]$  because

$$[\mathfrak{a}, a][\bar{\mathfrak{a}}, a] = [\mathfrak{a}\bar{\mathfrak{a}}, a^2] = [(a), a^2] = [\mathcal{O}, 1]$$

where  $\mathcal{O} = \mathbf{Z}[\xi]$  if  $[\mathfrak{a}, a] \in \mathcal{P}$ , and  $\mathcal{O} = \mathbf{Z}[1/p][\xi]$  if  $[\mathfrak{a}, a] \in \mathcal{P}_p$ .

According to the articles of Brown [4] and of Sjerve and Yang [11], a bijection exists between the elements of  $\mathcal{P}$  (resp.  $\mathcal{P}_p$ ) and the conjugacy classes of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z})$  (resp.  $\mathrm{Sp}(p-1, \mathbf{Z}[1/p])$ ). For the convenience of the reader, we will recall how this bijection is constructed. Let  $Y \in \mathrm{Sp}(p-1, \mathbf{Z})$  be of odd prime order  $p$ . Let  $\mathfrak{a}$  be a  $\mathbf{Z}[\xi]$ -module whose underlying  $\mathbf{Z}$ -module is  $\mathbf{Z}^{p-1}$ , with the action of  $\xi$  given by  $Y$ . Such a module is a fractional ideal in  $\mathbf{Q}(\xi)$ . Let

$$v_1 = (\alpha_1, \dots, \alpha_{p-1})^T \in \mathbf{Z}[\xi]^{p-1}$$

be an eigenvector of  $Y$  to the eigenvalue  $\xi$ , that is  $Yv_1 = \xi v_1$ . Then the module  $\mathfrak{a}$  we described above is the ideal

$$\mathfrak{a} = \mathbf{Z}\alpha_1 + \dots + \mathbf{Z}\alpha_{p-1} .$$

Since the eigenvector  $v_1$  is unique up to multiples, the ideal  $\mathfrak{a}$  is unique up to fractional equivalence. Let  $Y' = GYG^{-1}$  with  $G \in \mathrm{Sp}(p-1, \mathbf{Z})$ . Then  $w_1 = Gv_1$  is an eigenvector for  $Y'$  to the eigenvalue  $\xi$  and the corresponding ideal is also  $\mathfrak{a}$ . Let  $a = D^{-1}v_1^T J v_1$ , where  $D = p\xi^{(p+1)/2}/(\xi - 1)$ , then  $[\mathfrak{a}, a]$  is the equivalence class we are searching for. So we have defined a mapping, which sends the conjugacy class of  $Y$  to the equivalence class  $[\mathfrak{a}, a] \in \mathcal{P}$ . In [11] is shown that this mapping is a bijection. The construction for  $\mathrm{Sp}(p-1, \mathbf{Z}[1/p])$  is analogous.

Let  $C_0 := C_0(\mathbf{Z}[\xi])$  be the subgroup of the ideal class group  $\mathcal{C} = \mathcal{C}(\mathbf{Z}[\xi])$  given by

$$C_0 = \{\mathfrak{a} \in \mathcal{C} \mid \mathfrak{a}\bar{\mathfrak{a}} = (a), \quad a = \bar{a} \text{ for some } a \in \mathbf{Z}[\xi]\}.$$

Let  $C_p := \mathcal{C}(\mathbf{Z}[1/p][\xi])$  denote the ideal class group of the Dedekind domain  $\mathbf{Z}[1/p][\xi]$ . We define a subgroup  $C_{p0} := C_0(\mathbf{Z}[1/p][\xi])$  of  $C_p$ :

$$C_{p0} = \{\mathfrak{a}_p \in C_p \mid \mathfrak{a}_p \bar{\mathfrak{a}}_p = (a), \quad a = \bar{a} \text{ for some } a \in \mathbf{Z}[1/p][\xi]\}.$$

It follows directly from the definition, that for  $\mathfrak{a} \in C_0$  (resp.  $\mathfrak{a} \in C_{p0}$ ) holds  $[\mathfrak{a}, a] \in \mathcal{P}$  (resp.  $[\mathfrak{a}, a] \in \mathcal{P}_p$ ). But here we have  $a = \bar{a}$ , which was not requested in the definition of  $\mathcal{P}$  and  $\mathcal{P}_p$ . But for an equivalence class  $[\mathfrak{a}, a]$  we can always choose  $a$  such that  $a = \bar{a}$ . For a proof of this fact see [11].

Let  $U$  be the group of units in  $\mathbf{Z}[\xi]$  and  $U^+ := \{u \in U \mid u = \bar{u}\}$  the group of units in  $\mathbf{Z}[\xi + \xi^{-1}]$ . Let  $N: \mathbf{Q}(\xi) \rightarrow \mathbf{Q}(\xi + \xi^{-1})$ ,  $a \mapsto N(a) = a\bar{a}$ , be the norm mapping and  $N(U) := \{u\bar{u} = N(u) \mid u \in U\}$ . Let  $U_p$  be the group of units in  $\mathbf{Z}[1/p][\xi]$  and  $U_p^+ := \{u \in U_p \mid u = \bar{u}\}$ ,  $N(U_p) := \{u\bar{u} \mid u \in U_p\}$ . Clearly  $N(U) \subset U^+$ ,  $N(U_p) \subset U_p^+$ , and we can define the abelian groups  $U^+/N(U)$  and  $U_p^+/N(U_p)$ . It is well-known (see Washington [12]) that  $U_p = U \cdot \langle 1 - \xi \rangle$  where  $\langle 1 - \xi \rangle$  is the group generated by  $1 - \xi$ , and  $U_p^+ = U^+ \cdot \langle (1 - \xi)(1 - \xi^{-1}) \rangle$  where  $\langle (1 - \xi)(1 - \xi^{-1}) \rangle$  is the subgroup of  $\langle 1 - \xi \rangle$  generated by  $(1 - \xi)(1 - \xi^{-1})$ . Hence

$$(*) \quad [U_p^+ : N(U_p)] = [U^+ : N(U)] = 2^{(p-1)/2}$$

where the last equation is a consequence of the Dirichlet unit theorem.

According to the articles of Brown [4] and of Sjerve and Yang [11], there are short exact sequences of abelian groups

$$1 \longrightarrow U^+/N(U) \xrightarrow{\delta} \mathcal{P} \xrightarrow{\eta} C_0 \longrightarrow 1,$$

$$1 \longrightarrow U_p^+/N(U_p) \xrightarrow{\delta_p} \mathcal{P}_p \xrightarrow{\eta_p} C_{p0} \longrightarrow 1,$$

where  $\delta(uN(U)) = [\mathbf{Z}[\xi], u]$ ,  $\delta_p(uN(U)) = [\mathbf{Z}[1/p][\xi], u]$ ,  $\eta([\mathfrak{a}, a]) = [\mathfrak{a}]$  and  $\eta_p([\mathfrak{a}_p, a]) = [\mathfrak{a}_p]$ . Theorem 3 in the article of Sjerve and Yang [11] states that

the number of elements in  $\mathcal{P}$  is  $2^{(p-1)/2}h^-$ . Here  $h^- := h/h^+$  where  $h$  and  $h^+$  are the class numbers of  $\mathbf{Q}(\xi)$  and  $\mathbf{Q}(\xi + \xi^{-1})$  respectively. It follows from Proposition 7 in the article of Brown [4] that the cardinality of  $\mathcal{P}_p$  is  $2^{(p-1)/2}h^-$  too.

Now we will define homomorphisms  $\rho_1$ ,  $\rho$  and  $\rho_2$  such that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U^+/N(U) & \xrightarrow{\delta} & \mathcal{P} & \xrightarrow{\eta} & \mathcal{C}_0 & \longrightarrow & 1 \\ & & \rho_1 \downarrow & & \rho \downarrow & & \rho_2 \downarrow & & \\ 1 & \longrightarrow & U_p^+/N(U_p) & \xrightarrow{\delta_p} & \mathcal{P}_p & \xrightarrow{\eta_p} & \mathcal{C}_{p0} & \longrightarrow & 1 \end{array}$$

We define a homomorphism of abelian groups:

$$\begin{aligned} \rho_1: U^+/N(U) &\longrightarrow U_p^+/N(U_p) \\ uN(U) &\longmapsto uN(U_p). \end{aligned}$$

We have already seen that  $U_p = U \cdot \langle 1 - \xi \rangle$  where  $\langle 1 - \xi \rangle$  is the subgroup generated by  $1 - \xi$ . This implies that

$$N(U_p) = N(U) \cdot \langle (1 - \xi)(1 - \xi^{-1}) \rangle.$$

Let  $uN(U) \neq vN(U) \in U^+/N(U)$ , then  $uN(U_p) \neq vN(U_p)$ . Indeed, if  $uN(U_p) = vN(U_p)$ , then  $w \in N(U_p)$  exists with  $u = wv$ . But  $w \notin N(U)$  since  $uN(U) \neq vN(U)$ . On the other hand  $u = wv$  and  $u, v \in U^+$  imply that  $w \in U^+$ . But  $N(U_p) \not\subseteq U^+$  and this yields a contradiction. Therefore  $\rho_1$  is injective and  $\rho_1$  is an isomorphism since the equation (\*) holds.

Now we will define  $\rho_2: \mathcal{C}_0 \rightarrow \mathcal{C}_{p0}$ . Let  $\mathfrak{a} \subseteq \mathbf{Z}[\xi]$  be an ideal. Then we consider the ideal  $\mathfrak{a}_p \in \mathbf{Z}[1/p][\xi]$  generated by the elements  $\alpha z$  with  $\alpha \in \mathfrak{a}$ ,  $z \in \mathbf{Z}[1/p][\xi]$ . Since each  $z \in \mathbf{Z}[1/p][\xi]$  can be written as  $z = z'/p^r$ , where  $r \in \mathbf{N}$  and  $z' \in \mathbf{Z}[\xi]$ , we get  $\mathfrak{a}_p = \mathfrak{a}\mathbf{Z}[1/p][\xi]$ . So we can define a homomorphism

$$\begin{aligned} \rho_2: \mathcal{C}_0 &\longrightarrow \mathcal{C}_{p0} \\ [\mathfrak{a}] &\longmapsto [\mathfrak{a}_p]. \end{aligned}$$

Let  $[\mathfrak{a}], [\mathfrak{b}] \in \mathcal{C}_0$ ,  $[\mathfrak{a}] \neq [\mathfrak{b}]$ . Then  $[\mathfrak{a}_p] \neq [\mathfrak{b}_p]$ . Indeed, let  $\mathfrak{a}$  and  $\mathfrak{b}$  be representatives of  $[\mathfrak{a}]$  and  $[\mathfrak{b}]$  respectively. Then  $[\mathfrak{a}_p] = [\mathfrak{b}_p]$  would mean that there exist  $\lambda, \mu \in \mathbf{Z}[1/p][\xi]$  with  $\lambda\mathfrak{a}_p = \mu\mathfrak{b}_p$ . But then we would have  $[\mathfrak{a}] = [\mathfrak{b}]$ . Herewith  $\rho_2$  is injective and  $\rho_2$  is an isomorphism since  $|\mathcal{C}_0| = |\mathcal{C}_{p0}| = h^- < \infty$ .

Now it remains to define

$$\begin{aligned}\rho: \mathcal{P} &\longrightarrow \mathcal{P}_p \\ [\alpha, a] &\longmapsto [\alpha_p, a].\end{aligned}$$

Let  $\alpha\bar{\alpha} = (a)$ . Then  $\alpha_p\bar{\alpha}_p = (a)$ , a principal ideal in  $\mathbf{Z}[1/p][\xi]$ , and herewith  $\rho$  is well-defined. It follows directly from the definitions that  $\rho \circ \delta = \delta_p \circ \rho_1$  and  $\rho_2 \circ \eta = \eta_p \circ \rho$ . So the squares commute and, as a consequence of the five-lemma,  $\rho$  is an isomorphism.

Since  $\mathcal{P}$  and  $\mathcal{P}_p$  are isomorphic, each conjugacy class of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z}[1/p])$  contains an element of  $\mathrm{Sp}(p-1, \mathbf{Z})$ . This means that the isomorphism  $\rho: \mathcal{P} \rightarrow \mathcal{P}_p$  corresponds to mapping conjugacy classes of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z})$  to conjugacy classes of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z}[1/p])$ .

Now we will recall parts of the discussion in [11] that are important for our purposes. Let  $Y \in \mathrm{Sp}(p-1, \mathbf{Z})$  be of prime order  $p$  and let

$$v_1 = (\alpha_1, \dots, \alpha_{p-1})^T \in \mathbf{Z}[\xi]^{p-1}$$

be an eigenvector corresponding to the eigenvalue  $\xi$ , that is  $Yv_1 = \xi v_1$ . Let  $\alpha$  be the  $\mathbf{Z}$ -module generated by  $\alpha_1, \dots, \alpha_{p-1}$ . Then  $\alpha$  is an integral ideal in  $\mathbf{Z}[\xi]$  where the action of  $\xi$  on the  $\mathbf{Z}$ -module  $\alpha$  is given by  $Y$ . Let  $\gamma_j \in \mathrm{Gal}(\mathbf{Q}(\xi)/\mathbf{Q})$  with  $\gamma_j(\xi) = \xi^j$ ,  $j = 1, \dots, p-1$ , be an element of the Galois group. Then  $v_j = (\gamma_j(\alpha_1), \dots, \gamma_j(\alpha_{p-1}))^T$  is an eigenvector to the eigenvalue  $\xi^j$ . Now let  $a = D^{-1}v_1^T J \bar{v}_1$  where  $D = p\xi^{(p+1)/2}/(\xi-1)$ ,  $D = -\bar{D}$ . Then Sjerve and Yang showed that  $(\alpha, a)$  is a pair with  $\alpha\bar{\alpha} = (a)$ . Following the same procedure, we can find for a given matrix  $Y_p \in \mathrm{Sp}(p-1, \mathbf{Z}[1/p])$  an ideal  $\alpha_p \subseteq \mathbf{Z}[1/p][\xi]$  such that  $\alpha_p\bar{\alpha}_p = (a)$ .

The sign of the invariant subspace corresponding to the eigenvalues  $\xi^j, \xi^{-j}$  of  $Y$  is

$$\mathrm{sign}(V_j) = \mathrm{sign} \mathrm{Im}(q(v_j, \bar{v}_j)) = \mathrm{sign}(-i\gamma_j(Da))$$

where the sign of  $z \in \mathbf{Z}[\xi + \xi^{-1}]$  is the sign of  $\iota(z)$  for the real embedding  $\iota$  of  $\mathbf{Z}[\xi + \xi^{-1}]$  with  $\iota(\xi + \xi^{-1}) = e^{i2\pi/p} + e^{-i2\pi/p}$ . Now we see that  $\psi$  is surjective if and only if

$$\psi': \{a \in \mathbf{Z}[\xi] \mid \exists \alpha \text{ with } (\alpha, a) \in P\} \longrightarrow (\mathbf{Z}/2\mathbf{Z})^{(p-1)/2}$$

with

$$a \longmapsto (\mathrm{sign}(\gamma_1(a)), \dots, \mathrm{sign}(\gamma_{(p-1)/2}(a)))$$

is surjective. We call  $a \in \mathbf{Q}(\xi)$  a Hermitian square if  $x \in \mathbf{Q}(\xi)$  exists such that  $x\bar{x} = a$ . Now we use Lemma 2.3 in the article of Alexander, Conner, Hamrick and Vick [2]. We repeat the statement of this lemma.

LEMMA 1.7. *Let  $a \neq 0$  be a  $\mathbf{Z}[1/p][\xi]$ -ideal with  $a\bar{a} = a\mathbf{Z}[1/p][\xi]$ . Then  $a$  is a Hermitian square if and only if it is positive in every ordering of  $\mathbf{Q}(\xi + \xi^{-1})$ .*

This implies that

$$\psi'_p: \{a \in \mathbf{Z}[1/p][\xi] \mid \exists \alpha \text{ with } (a, \alpha) \in P_p\} \longrightarrow (\mathbf{Z}/2\mathbf{Z})^{(p-1)/2}$$

with

$$a \longmapsto (\text{sign}(\gamma_1(a)), \dots, \text{sign}(\gamma_{(p-1)/2}(a)))$$

is surjective. But then  $\psi_p$  is surjective and therefore  $\psi$  is surjective too. Herewith we have completed the proof of Theorem 1.2.

### 1.2.3 CONCERNING LEMMA 1.7

We give here some more information on Lemma 1.7 since it is crucial in the proof of Theorem 1.2 and only a sketch of a proof is given in [2].

One direction is obvious. To see that the lemma is true, it is necessary to study Hilbert symbols in  $\mathbf{Q}(\xi + \xi^{-1})$ . We define  $\sigma := \xi + \xi^{-1} - 2$ . Then  $\mathbf{Q}(\xi) = \mathbf{Q}(\xi + \xi^{-1})(\sqrt{\sigma})$ . Let  $\mathfrak{p}$  be a prime in  $\mathbf{Q}(\xi + \xi^{-1})$ . A fundamental property of the Hilbert symbol is

$$\left(\frac{a, \sigma}{\mathfrak{p}}\right) = 1 \iff a \text{ is a norm of the extension } \mathbf{Q}(\xi)/\mathbf{Q}(\xi + \xi^{-1}).$$

A proof of this property can be found in the books [9] and [10] of Neukirch. So  $a$  is a Hermitian square if and only if

$$\left(\frac{a, \sigma}{\mathfrak{p}}\right) = 1 \text{ for all primes, finite or infinite, in } \mathbf{Q}(\xi + \xi^{-1}).$$

We first consider the infinite primes. Therefore we use the connection of the Hilbert symbol with the norm residue symbol (see [9] and [10]). For infinite primes we have the norm residue symbol for  $\mathbf{C}/\mathbf{R}$

$$(\cdot, \mathbf{C}/\mathbf{R}): \mathbf{R}^* \longrightarrow \text{Gal}(\mathbf{C}/\mathbf{R})$$

defined by

$$(a, \mathbf{C}/\mathbf{R})\sqrt{-1} = \sqrt{-1}^{\text{sign}(a)}.$$

The kernel of this homomorphism is

$$\mathbf{R}_{>0} = N_{\mathbf{C}/\mathbf{R}}(\mathbf{C}^*) = \{z\bar{z} \mid z \in \mathbf{C}^*\}$$

where  $\mathbf{C}^*$  and  $\mathbf{R}^*$  denote the multiplicative subgroup of  $\mathbf{C}$  and  $\mathbf{R}$  respectively. So the positivity required in Lemma 1.7 implies that the Hilbert symbol is 1 at infinite primes. It remains to consider the finite primes. The Hilbert symbol is also 1 at the inert primes because of the following lemma.



LEMMA 1.8. *If  $a \in \mathbf{Q}(\xi + \xi^{-1})$ , then there is a fractional ideal  $\mathfrak{a} \subset \mathbf{Q}(\xi)$  with  $\mathfrak{a}\bar{\mathfrak{a}} = a\mathbf{Z}[\xi]$  if and only if at every inert prime  $\mathfrak{p} \subset \mathbf{Z}[\xi + \xi^{-1}]$  we have*

$$\left(\frac{a, \sigma}{\mathfrak{p}}\right) = 1.$$

*Proof.* See [1].  $\square$

If  $\mathfrak{p}$  is a prime in  $\mathbf{Q}(\xi + \xi^{-1})$  that splits, then the Hilbert symbol

$$\left(\frac{a, \sigma}{\mathfrak{p}}\right) = 1$$

(see [1]). So it remains to consider the ramified primes in  $\mathbf{Q}(\xi + \xi^{-1})$ . But the only prime that ramifies is  $\sigma\mathbf{Z}[\xi + \xi^{-1}]$ . Then, by the reciprocity law of Hilbert symbols (see [9]), the Hilbert symbol at this prime is 1.

This proves Lemma 1.7.

#### 1.2.4 AN INTERESTING REMARK

Let  $U$  be the group of units in  $\mathbf{Z}[\xi]$  and  $U^+ = \{u \in U \mid u = \bar{u}\}$ . Let  $u \in U^+ \setminus N(U)$  where  $N$  is the norm map. Then  $[\mathfrak{a}, a] \in \mathcal{P}$  implies that  $[\mathfrak{a}, ua] \in \mathcal{P}$  and  $[\mathfrak{a}, a] \neq [\mathfrak{a}, ua]$ . Let  $Y$  be a representative of the conjugacy class of matrices corresponding to  $[\mathfrak{a}, a]$ . We have seen that the  $\text{sign}(V_j)$  of  $Y$  is given by  $a$ . Let us fix the ideal  $\mathfrak{a}$ . The question that arises now is if the restriction of  $\psi$  to the conjugacy classes of matrices corresponding to  $[\mathfrak{a}, ua]$ , where  $u$  is as above, is surjective. But this restriction is not surjective for each prime. Let  $h$  and  $h^+$  be the class numbers of  $\mathbf{Q}(\xi)$  and  $\mathbf{Q}(\xi + \xi^{-1})$  respectively. Then  $h^- = h/h^+$ . Let  $C$  denote the group of cyclotomic units in  $\mathbf{Q}(\xi)$  and let  $C^+ = C \cap \mathbf{Z}[\xi + \xi^{-1}]$ . It is known that  $[\mathbf{Z}[\xi + \xi^{-1}]^* : C^+] = h^+$ . We can find in the article of Garbanati [8] that  $h^-$  is odd if and only if  $C^+$  contains units of all signatures, which means that every totally positive unit in  $C^+$  is the square of a unit of  $C$ . So in case  $h^-$  is odd,

$$\begin{aligned} \omega: U^+ \setminus N(U) &\longrightarrow (\mathbf{Z}/2\mathbf{Z})^{(p-1)/2} \\ u &\longmapsto (\text{sign}(\gamma_1(u)), \dots, \text{sign}(\gamma_{(p-1)/2}(u))) \end{aligned}$$

is surjective, and this implies the surjectivity of  $\psi'$ . However it may be possible that  $\mathbf{Z}[\xi + \xi^{-1}]^*$  contains units of all signatures even if  $C^+$  does not. This can only happen if  $h^+$  is even and then we do not know if  $\omega$  is surjective. If  $h^-$  is even and  $h^+$  is odd, we have no surjectivity of  $\omega$ , and the restriction of  $\psi'$  to  $\{a \in \mathbf{Z}[\xi] \mid (\mathfrak{a}, a) \in P\}$  for a fixed ideal  $\mathfrak{a}$  is not surjective either. This happens for example for the primes 29 and 113.