

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 47 (2001)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CIRCULANT MODULAR HADAMARD MATRICES  
**Autor:** Eliahou, Shalom / Kervaire, Michel  
**Kapitel:** 3. Circulant modular Hadamard matrices of type 2  
**DOI:** <https://doi.org/10.5169/seals-65430>

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Equivalently, this “remainder”  $R(z)$  can be written

$$(11) \quad R(z) = 2 \sum_{\nu=1}^{\frac{p-1}{2}} (z^{4\nu} + z^{-4\nu}) + \left\{ \sum_{\nu=1}^p (z^{2\nu-1} + z^{-(2\nu-1)}) + z^p + z^{-p} \right\} \varepsilon_0 \varepsilon_1.$$

The (periodic) correlations of  $H(z)$  in degrees  $\equiv 2 \pmod{4}$  are strictly zero. This includes in particular the correlation of degree  $2p$ . Hence, the modular Hadamard matrix associated with the sequence (polynomial) of the Theorem is indeed of type 1 as asserted. The correlations in degrees  $\equiv 0 \pmod{4}$  are  $2(p-1)$ . Note that the correlation in degree  $p$  is  $2(p-1) \varepsilon_0 \varepsilon_1$  because  $z^p + z^{-p}$  also appears in the sum  $\sum_{\nu=1}^p (z^{2\nu-1} + z^{-(2\nu-1)})$  for  $\nu = \frac{p+1}{2}$ .

REMARK. It seems probable, from computer-assisted experimentation, that  $p-1$  may be the maximum modulus for a modular circulant Hadamard matrix of type 1 and size  $4p$ . However, the power of 2 dividing  $p-1$  is certainly not always maximal as the power of 2 dividing the modulus of a modular CHM of type 1 and size  $4p$ . There are many values of  $p$  (where  $p$  is prime and satisfies  $p \equiv 9 \pmod{16}$ ) for which a variant of the formula for  $H(z)$  in the above Theorem yields a 16-modular CHM. The first few such values of  $p$  are  $p = 73, 89, 233, \dots$ . On the other hand, it seems for example that indeed no 16-modular, type 1 CHM of size  $4p$  exists for  $p = 41$ .

We hope to come back on the general question of 16-modular circulant Hadamard matrices of type 1 in a future publication.

### 3. CIRCULANT MODULAR HADAMARD MATRICES OF TYPE 2

In this section we produce circulant modular Hadamard matrices of type 2 and size  $n = 2(q+1)$ , where  $q$  is an arbitrary odd prime power. The existence of such objects is a corollary of a theorem from the 1971 paper [DGS].

We are grateful to Roland Bacher for valuable discussions about some unpublished work of his which helped in obtaining the following result.

**THEOREM 2.** *For every  $n = 2(q+1)$ , where  $q$  is an odd prime power, there exists a binary sequence  $X = (x_0, \dots, x_{n-1})$  with  $x_i = \pm 1$  for all  $i$  ( $0 \leq i \leq n-1$ ), such that  $\gamma_k(X) = 0$  for all  $k \neq 0, \frac{n}{2}$ . In other words,  $\text{circ}(X)$  is a circulant modular Hadamard matrix of type 2 and size  $n$ .*

*Proof.* Set  $x_{\frac{n}{2}} = x_0 = 1$  and  $x_{\frac{n}{2}+i} = -x_i$  for all  $i = 1, 2, \dots, \frac{n}{2} - 1$ . The sequence  $X = (x_1, x_2, \dots, x_{n-1})$  is therefore determined by its subsequence  $Y = (x_1, x_2, \dots, x_{\frac{n}{2}-1})$ .

We have  $\gamma_0(X) = n$ ,  $\gamma_{\frac{n}{2}}(X) = 4 - n$ , and

$$\gamma_k(X) = 2(\alpha_k(Y) - \alpha_{\frac{n}{2}-k}(Y))$$

for all  $k = 1, 2, \dots, \frac{n}{2} - 1$  as easily checked, where  $\alpha_k$  is the  $k$ th *aperiodic* correlation coefficient. Of course,  $\gamma_{n-k}(X) = \gamma_k(X)$  for all  $k = 1, 2, \dots, \frac{n}{2} - 1$ .

In order to prove the theorem, it therefore suffices to exhibit a binary sequence  $Y = (x_1, x_2, \dots, x_{\frac{n}{2}-1})$  of length  $\frac{n}{2} - 1 = q$ , satisfying the equation  $\alpha_k(Y) - \alpha_{\frac{n}{2}-k}(Y) = 0$  for every  $k = 1, 2, \dots, \frac{n}{2} - 1$ .

For this purpose, we recall the notion of a *negacyclic* matrix, introduced by Delsarte, Goethals and Seidel in their paper [DGS].

By definition it is simply a matrix of the form

$$\begin{pmatrix} u_0 & u_1 & \dots & \dots & u_r \\ -u_r & u_0 & u_1 & \dots & u_{r-1} \\ -u_{r-1} & -u_r & u_0 & \ddots & u_{r-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -u_1 & -u_2 & \dots & -u_r & u_0 \end{pmatrix}$$

which we will denote by  $NC(u_0, u_1, \dots, u_r)$ .

Explicitly, the entries  $c_{i,j}$  of the matrix  $NC(u_0, u_1, \dots, u_r)$  are

$$c_{i,j} = \begin{cases} u_{j-i} & \text{if } 0 \leq i \leq j \leq r, \\ -u_{r-i+j+1} & \text{if } 0 \leq j < i \leq r. \end{cases}$$

It is very easy to see that the binary sequence  $Y = (x_1, x_2, \dots, x_{\frac{n}{2}-1})$  satisfies  $\alpha_k(Y) - \alpha_{\frac{n}{2}-k}(Y) = 0$  for every  $k = 1, 2, \dots, \frac{n}{2} - 1$  if and only if the negacyclic matrix  $C = NC(0, x_1, \dots, x_{\frac{n}{2}-1})$  is a conference matrix, that is if  $C \cdot C^t = (\frac{n}{2} - 1)I$ .

Now, Delsarte, Goethals and Seidel have explicitly constructed negacyclic conference matrices of every size of the form  $q + 1$ , where  $q = p^f$  with  $p$  an odd prime and  $f$  a positive integer, in Section 7 of [DGS]. These negacyclic conference matrices are equivalent to the usual Paley conference matrices based on the quadratic character  $\chi: \mathbf{F}_q^* \rightarrow \{\pm 1\}$  of the finite field  $\mathbf{F}_q$ .  $\square$

NOTE. After having submitted the present paper for publication, we came across the Thèse d'Habilitation of Philippe Langevin (Toulon). There, a concept which is closely related to our type 2 sequences is studied. P. Langevin uses the terminology "almost perfect sequences" and his treatment also relies on [DGS].

Thus, we now find it preferable to drop the type 1 / type 2 terminology and rather call *enhanced modular* the modular matrices of type 1. We intend to use this new designation in future publications on the subject.

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(Reçu le 18 juillet 2000)

Shalom Eliahou

Département de Mathématiques  
LMPA Joseph Liouville  
Université du Littoral Côte d'Opale  
Bâtiment Poincaré  
50, rue Ferdinand Buisson, B.P. 699  
F-62228 Calais  
France  
*e-mail*: eliahou@lmpa.univ-littoral.fr

Michel Kervaire

Département de Mathématiques  
Université de Genève  
2-4, rue du Lièvre  
B.P. 240  
CH-1211 Genève 24  
Suisse  
*e-mail*: Michel.Kervaire@math.unige.ch