4. Length estimates for systoles

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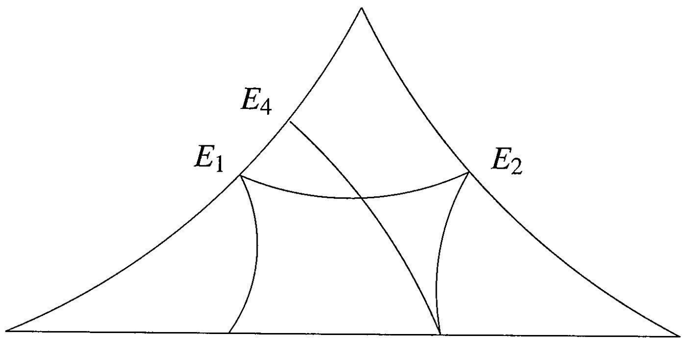
b) *The arcs* $\widetilde{\zeta}_1$ *and* $\widetilde{\zeta}_4$ *do not intersect.*



In this case either the arc $\widetilde{\zeta}_1$ intersects the arc $\widetilde{\zeta}_3$ or the arc $\widetilde{\zeta}_4$ intersects the arc $\widetilde{\zeta}_2$. Assume that the second case holds.

Let again E_1, E_2 be the endpoints of $\widetilde{\zeta}_2$ where E_1 lies on the edge b and let E_4 be the endpoint of the arc $\widetilde{\zeta}_4$ on the edge b. Since $\widetilde{\zeta}_4$ meets b orthogonally at E_4 and has its second endpoint E_3 on the side a, the angle at E_4 of the triangle with vertices E_1, E_4, E_2 is strictly bigger than $\frac{\pi}{2}$. This means that the distance between E_2 and E_4 is smaller than the length of the arc $\widetilde{\zeta}_2$ and therefore the length of $\widetilde{\zeta}$ is bigger than the circumference of the triangle with vertices E_2, E_3, E_4. In particular, this length is bigger than the length of the A-orbit $\widetilde{\gamma}_1$.

This completes the proof of our lemma. □

As an immediate corollary of Lemma 3.6 and Lemma 3.5 we obtain

COROLLARY 3.7. *A C-orbit in $\triangle$ does not lift to a systole on S.*

4. LENGTH ESTIMATES FOR SYSTOLES

In this section we complete the geometric description of the systoles of a simple triangle surface and its associated ideal surface. As a consequence we obtain that a simple triangle surface which is different from one of the three surfaces listed in the introduction is not maximal.

We resume the assumptions and notations from Section 3. Our goal is to describe all B-orbits in the equilateral triangle $\triangle$ with angle π/p or in an ideal triangle $\triangle_\infty$ which lift to a systole on a simple triangle surface S or its associated ideal surface S_∞. For this it is convenient to consider any piecewise geodesic α in $\triangle$ with the following properties:

a) There is a pair e_1, e_2 of sides of $\triangle$ which is connected by at most one subarc of α.

b) If e_3 is the third side of $\triangle$ then the subcurves α_1, α_2 of α which contain all arcs of α joining e_1, e_2 to e_3 are connected and either $\alpha = \alpha_1\alpha_2$ or $\alpha_1\alpha_2$ is not connected.

We call such a curve *irreducible*. A B-orbit $\widetilde{\gamma}$ which is irreducible in this sense and with the additional property that there is a pair of sides of $\triangle$ which is not connected by any geodesic segment of $\widetilde{\gamma}$ will be called a B_0*-orbit.* An irreducible B-orbit which is not a B_0-orbit will be called a B_1*-orbit.* In the same way we define irreducible B_0-orbits and B_1-orbits in the ideal triangle $\triangle_\infty$.

A *lift* to S/Γ of an irreducible curve α in $\triangle$ is an admissible closed piecewise geodesic in $S/\Gamma\backslash\{\widehat{A}, \widehat{B}, \widehat{0}\}$ whose trace is invariant under the natural isometry $\widehat{\Psi}$ of order 2 of S/Γ exchanging the two triangles and which projects to α. Call two irreducible curves α, β in $\triangle$ *homotopic* if there are lifts of β and α to S/Γ which are freely homotopic in $S/\Gamma - \{\widehat{A}, \widehat{B}, \widehat{0}\}$.

The remark after Lemma 3.5 shows that a B-orbit in $\triangle$ is irreducible in the above sense if and only if its lift to $S/\Gamma - \{\widehat{0}, \widehat{A}, \widehat{B}\}$ is irreducible in the sense of Section 3. Thus we obtain from the results in Section 3.

COROLLARY 4.1. *A B-orbit in $\triangle$ or $\triangle_\infty$ which lifts to a systole on S or S_∞ is irreducible.*

For the description of all B-orbits in $\triangle$ which lift to a systole of a simple triangle surface we use a length comparison argument. Namely, observe that we can talk about homotopic irreducible arcs in nonisometric hyperbolic triangles in an obvious way. We have.

LEMMA 4.2. *Let $q > p \geq 5$ and let $\triangle, \triangle'$ be equilateral triangles with angles $\pi/p, \pi/q$ respectively. Let γ, γ' be two homotopic B-orbits in $\triangle, \triangle'$. Then the length of γ is smaller than the length of γ'.*

Proof. For $t < \pi/3$ denote by T_t the equilateral hyperbolic triangle with angle t. Since a B-orbit is the shortest curve in its homotopy class it suffices to show the following: If $t < t_0 < \pi/3$ and if $\gamma \subset T_{t_0}$ is any B-orbit, then every admissible curve in T_t which is homotopic to γ is longer than γ.

But this follows simply from the fact that for $t < t_0$ the triangle T_{t_0} can be isometrically embedded into the triangle T_t (see [I]). More precisely, the center of the triangle T_t is the unique point in T_t which has the same distance to each of the vertices of T_t. There is an (essentially unique) isometric embedding of T_{t_0} into T_t which maps the center of T_{t_0} to the center of T_t and such that each geodesic in T_t which connects the center to one of the vertices passes through a vertex of T_{t_0}. Map T_{t_0} onto T_t by a diffeomorphism which maps each geodesic γ through the center to itself and scales the parametrization by the proportionality factor $\text{length}(\gamma \cap T_t)/\text{length}(\gamma \cap T_{t_0})$. This map strictly increases the length of nontrivial curves in T_{t_0}. From this the lemma is immediate. □

Let again Ω be a fundamental $2p$-gon, let $k \in [2,(p+1)/2]$ and let $S = S(p;k)$ be a simple triangle surface. The side pairings for Ω which induce the surface S define a collection of p simple closed geodesics on S which are invariant under the action of the basic group Γ. Each of these geodesics is freely homotopic to the projection to S of a geodesic arc in Ω connecting the midpoint of the side $2i+1$ to the midpoint of the side $2i+2k$. Their projection to S/Γ is the lift of an irreducible B_0-orbit $\widetilde{\gamma}_0$ which can be described as follows.

a) $\widetilde{\gamma}_0$ has one endpoint on the edge opposite to a vertex $\widetilde{0}$ which is the only collision point with this edge.

b) There are k collisions with the edge joining $\widetilde{0}$ to a second vertex $\widetilde{A}$ and $k-1$ collisions with the edge joining $\widetilde{0}$ to the third vertex $\widetilde{B}$ for some $k \in [2,p/2]$.

We call a B_0-orbit $\widetilde{\gamma}$ with properties a) and b) for an arbitrary $k \leq p/2$ a *side pairing orbit.* With this notation, every minimal B_0-orbit is a side pairing orbit. Moreover a side pairing orbit is determined up to isometries of $\triangle$ by the number of its geodesic segments, or, equivalently, by the number of its collision points with the boundary of $\triangle$. For a simple triangle surface S there are at most three different liftable side pairing orbits (compare Section 2).

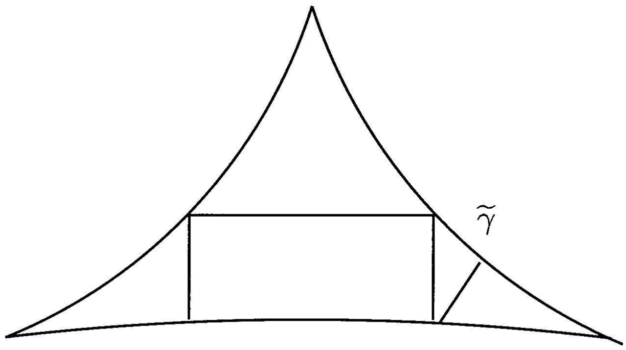
Using Lemma 4.2 and a comparison argument we can now estimate the length of a large family of irreducible B-orbits.

LEMMA 4.3. *Let $\widetilde{\eta}$ be an irreducible B-orbit. Assume that either*

1. *$\widetilde{\eta}$ is a B_1-orbit with at least 5 collisions with the boundary or*
2. *$\widetilde{\eta}$ is a B_0-orbit which is not a side pairing orbit and has at least 6 collisions with the boundary.*

Then a lift of $\widetilde{\eta}$ to $S/\Gamma - \{\widehat{A},\widehat{B},\widehat{0}\}$ is longer than a systole on S.

Proof. By definition, a B_1-orbit contains at least 3 geodesic arcs. Up to isometries of $\triangle$ there is a unique B_1-orbit $\widetilde{\gamma}$ consisting of exactly 4 arcs.



This orbit admits a subarc which is homotopic to a side pairing orbit with 3 segments. In particular, if $S = S(p;k)$ admits a liftable side pairing orbit which consists of at most three segments, then this side pairing orbit is homotopic to a proper subarc of $\widetilde{\gamma}$ and therefore a lift of $\widetilde{\gamma}$ to S/Γ is longer than a systole on S.

Lemma 2.2 shows that for $p \leq 9$ every simple triangle surface of genus $\frac{p-1}{2}$ is isometric to a surface $S(p;m)$ for $m = 2$ or $m = 3$ and hence admits a liftable side pairing orbit which consists of at most 3 segments.

On the other hand, an explicit computation (using Maple or Mathematica) shows that for $p = 11$ the length of $\widetilde{\gamma}$ is bigger than 3 arccosh $\frac{3}{2}$. Thus by Lemma 3.2, Lemma 4.2 and the above, a lift of $\widetilde{\gamma}$ to $S/\Gamma - \{\widehat{A}, \widehat{B}, \widehat{0}\}$ is longer than a systole on $S(p;k)$.

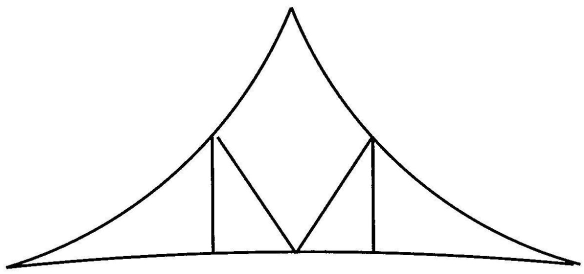
Since every B_1-orbit $\widetilde{\eta}$ with at least 5 collisions with the boundary admits a subarc which is homotopic to $\widetilde{\gamma}$, our statement for B_1-orbits follows.

Let $\widetilde{\eta}$ be a B_0-orbit which is not a side pairing orbit and has at least 6 collisions with the boundary. Denote by C the vertex of $\triangle$ whose adjacent sides are not connected by any subarc of $\widetilde{\gamma}$. Then $\widetilde{\gamma}$ contains a subarc which consists of two segments and connects the sides adjacent to C. If we replace this arc by a single geodesic segment, then we obtain a shorter curve which contains a subcurve homotopic to the B_1-orbit $\widetilde{\gamma}$ above. Thus the statement for B_0-orbits follows once again from the length estimate for $\widetilde{\gamma}$. □

COROLLARY 4.4. *Every systole on a simple triangle surface is either a lift of the A-orbit* $\widetilde{\gamma}_1$ *in* $\triangle$ *or a lift of a side pairing orbit on* $\triangle$.

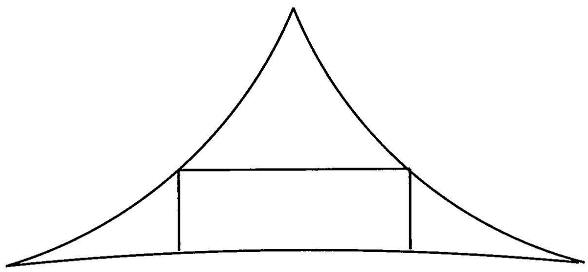
Proof. By Lemma 4.3, a B-orbit $\widetilde{\eta}$ which is not a side pairing orbit can only lift to a systole if either

1) $\widetilde{\eta}$ is a B_0-orbit with exactly 5 collisions with the boundary



or

2) $\widetilde{\eta}$ is a B_1-orbit with exactly 4 collisions with the boundary.



Consider first an orbit $\widetilde{\eta}$ as in 1) above. Assume that $\widetilde{\eta}$ lifts to a closed geodesic on the surface $S(p;k)$. The lifts of $\widetilde{\eta}$ then define piecewise geodesics in the fundamental $2p$-gon Ω.

Choose such a piecewise geodesic η with the property that the center of Ω corresponds to a vertex of $\triangle$ whose adjacent sides are connected by an arc of $\widetilde{\eta}$. Then η consists of two components η_1, η_2. After a suitable numbering of the edges of Ω we may assume that η_1 connects the edge 1 to the edge 6 and that η_2 connects the edge $6-2k+1$ to the edge $6-2k+2$ where $k \geq 2$ is such that $S = S(p;k)$.

Since η projects to a closed geodesic on $S(p;k)$ we have $6-4k+3 \equiv 1 \mod 2p$ and therefore $4-2k \equiv 0 \mod p$. Since p is odd and $k \leq p-1$ this is only possible if $k=2$. But then there is a liftable side pairing orbit of $S(p;k)$ which consists of 2 segments and is shorter than $\widetilde{\eta}$.

A similar purely combinatorial argument shows that an orbit $\widetilde{\eta}$ as in 2) above is not liftable to any simple triangle surface. This shows the lemma. □

Now we are ready to show

PROPOSITION 4.5.

1) *For* $3 \leq k \leq 5$ *the surface* $S(k(k-1)+1;k)$ *and its associated ideal surface has* $3k(k-1)+3$ *systoles. These systoles are the images of a single simple closed geodesic under the action of the isometry group of* $S(k(k-1)+1;k)$.

2) *A simple triangle surface* S *which is not isometric to one of the surfaces* $S(k(k-1)+1;k)(3 \leq k \leq 5)$ *is not maximal, neither is the ideal surface associated to* S.

Proof. Let $p = 2g+1$ for an arbitrary $g \geq 2$ and let S be a simple triangle surface of genus g.

Recall that there are numbers $k(0), k(A), k(B) \geq 2$ such that the side-pairings of the $2p$-gon Ω defining S with base-point $0, A, B$ identify the edge 1 with the edge $2k(0), 2k(A), 2k(B)$.

Let $k_0 = \min\{k(0), k(A), k(B)\}$ and assume (via renaming) that $k_0 = k(0)$. The projection to S of the geodesic arc $\tilde{\gamma}_0$ in Ω which connects the edge 1 to the edge $2k_0$ and is orthogonal to both edges is then a simple closed geodesic γ_0 in S whose length we denote by ℓ_0.

Corollary 4.4 shows that there are only two possibilities for a systole γ on S.

1) γ is a lift γ_1 of length ℓ_1 of the A-orbit $\tilde{\gamma}_1$ on $\triangle$ of period 3.

2) γ is the image under an isometry of S of the geodesic γ_0 of length ℓ_0.

Consider a surface $S = S(p;k)$ as in Lemma 2.2 which admits a cyclic group Σ of order 3 of isometries normalizing the basic group Γ. If ℓ_0 is smaller than ℓ_1 then S admits $3p = 6g+3$ systoles which are just the lifts of the unique liftable side pairing orbit for S. We claim that this is the case if and only if $S = S(7;3)$ or $S = S(13;4)$ or $S = S(21;5)$.

To see this, recall from Lemma 2.2 that each such surface with these additional symmetries is of the form $S = S(p;k)$ for some $k \geq 3$ and a divisor $p > k$ of $k(k-1)+1$. The unique liftable side pairing orbit for $S(p;k)$ consists of $\min\{k, p-k+1\}$ segments. However, explicit computation shows that a side pairing orbit with 6 segments in an equilateral triangle with angle $\pi/15$ is longer than the upper bound 3 arccosh $\frac{3}{2}$ for ℓ_1. Together with Lemma 4.2 this shows that if $S(p;k)$ is such that $\ell_0 \leq \ell_1$ then either $p \leq 13$ or $\min\{k, p-k+1\} \leq 5$.

The surfaces $S(7;3)$ and $S(13;4)$ are such surfaces $S(p;k)$ with $p \leq 13$. Any further example corresponds to a pair of numbers (p,k) such that $k < p \leq 13$ and that moreover p is a *proper* divisor of $k(k-1)+1$.

However the only pairs of this kind are $(13,10)$ and $(7,5)$ and we find once again our surfaces $S(13;10)=S(13;4)$ and $S(7;5)=S(7;3)$.

Next we look for surfaces $S(p;k)$ as above with $\min\{k,p-k+1\}\leq 5$ and such that $p>\min\{k,14\}$ is a divisor of $k(k-1)+1$. Write $m=p-k$ and assume that $m\leq 4$ and that $p=k+m$ divides $k(k-1)+1=(p-m)(p-m-1)+1=p(p-2m-1)+m(m+1)+1$. Then p also divides $m(m+1)+1$, and since we assumed that $p\geq 15$ we just obtain the surface $S(21;17)=S(21;5)$ as a solution.

In other words, if $\ell_0\leq\ell_1$ and if $S(p;k)$ admits a cyclic group of order 3 of isometries normalizing the basic group Γ then S is one of the surfaces $S(7;3)$, $S(13;4)$ and $S(21;5)$. Explicit computation now shows that for these surfaces we indeed have $\ell_0<\ell_1$.

Schmutz observed in [S1] that a closed hyperbolic surface S of genus g can only be maximal if S has at least $6g-5$ systoles. Using this fundamental fact, the proof of our proposition can now be reduced to the above discussion by distinguishing the following 4 cases.

i) $\ell_1<\ell_0$.

Then only lifts of the A-orbit $\widetilde{\gamma}_1$ can be systoles of S. If g is the genus of S then there are $p=2g+1$ systoles, and S is not maximal.

ii) $S=S(p;2)$ for some $p\geq 5$.

The surface $S(p;2)$ admits a liftable side pairing orbit $\widetilde{\gamma}_0$ which consists of 2 segments and hence is shorter than the orbit $\widetilde{\gamma}_1$ from Lemma 3.2. Moreover it admits a cyclic group Σ of order 2 of isometries which commutes with the basic group Γ. The action of Σ on the sphere S/Γ does not leave the trace of a lift of the side pairing orbit $\widetilde{\gamma}_0$ invariant. Thus $S(p;2)$ has exactly $2p=4g+2$ systoles and can only be maximal if either $g=2$ or $g=3$. However an explicit analysis of the surfaces $S(5;2)$ and $S(7;2)$ shows that these surfaces are not maximal (this fact was already established by Schmutz [S1]).

iii) $S\notin\{S(k(k-1)+1;k)\mid k\geq 2\}\cup\{S(p;2)\mid p\geq 5\}$ and $\ell_0\leq\ell_1$.

Then if $k_0=k(0)$ we have $k(A)>k_0, k(B)>k_0$ and therefore there are at most $p=2g+1$ systoles which are lifts of a side pairing orbit in $\triangle$. If $\ell_0<\ell_1$ then these are the only systoles. In the case $\ell_1=\ell_0$ (which does not occur if the genus g of S is 2 or 3) there are $4g+2$ systoles. The surface S is not maximal.

iv) $k\in\{3,4,5\}$ and $S=S(k(k-1)+1;k)$.

Then the length ℓ_0 of γ_0 is smaller than ℓ_1 and there are $3p=6g+3$ systoles which are the images of the geodesic γ_0 under the isometry group of S. In particular, the cardinality of the quotient of the isometry group of S under the subgroup fixing a given systole equals $6g+3$.

To complete the proof of our proposition we have to investigate the ideal surfaces S_∞ associated to simple triangle surfaces $S(p;k)$. The above considerations are equally valid for these surfaces and show that S_∞ has more than $4g+4$ systoles if and only if p divides $k(k-1)+1$ and if the length ℓ_0 of a lift of a side pairing orbit for S_∞ is not bigger than 6 arccosh $\frac{3}{2}$. An explicit computation shows as before that this is the case if and only if S_∞ is associated to one of the surfaces $S(7;3), S(13;4), S(21;5)$. □

5. Proof of the theorem

Using the notation of Lemma 2.2, our goal is to show that the triangle surfaces $S(7;3), S(13;4), S(21;5)$ and their associated ideal surfaces are maximal. Following Schmutz [S1], for this it is enough to show that for each of these surfaces S the Teichmüller space is parametrized in a neighborhood of S by the lengths of those closed geodesics which are freely homotopic to a systole on S.

Let for the moment $p \geq 5$ be an arbitrary odd number and let $k \in \{2,\dots,p-1\}$ be such that k and $k-1$ are prime to p. Write $g=(p-1)/2$. As in the introduction let $\mathcal{T}_{g,3}$ be the Teichmüller space of surfaces of genus g with 3 punctures. Let $S=S(p;k)$ and let S_∞ be the ideal surface associated to S. The basic group Γ of orientation preserving isometries of S acts as a group of isometries on the surface S_∞.

It will be useful to give a geometric description of S_∞. For this let $\triangle_\infty$ be an ideal triangle in $\mathbf{H}^2$ and let $T \subset \triangle_\infty$ be the finite equilateral triangle inscribed in $\triangle_\infty$ which is invariant under all isometries of $\triangle_\infty$. The vertices of T determine a distinguished point on each side of $\triangle_\infty$.

There is a unique way to glue $2p$ copies of $\triangle_\infty$ to a disc A with one puncture in its interior and $2p$ punctures on the boundary in such a way that the glueing maps identify the distinguished points on the sides of $\triangle_\infty$. The boundary of A then consists of $2p$ geodesic lines. Each of the triangles which makes up A contains exactly one of these boundary geodesics. We number the boundary geodesics in counter clockwise order and glue the $2i+1$-th geodesic to the $2i+2k$-th geodesic by an orientation reversing isometry which identifies the distinguished points on these geodesics. The resulting surface is the ideal surface S_∞ associated to S. Notice that S_∞ admits a canonical triangulation into ideal triangles which corresponds to the canonical triangulation of S.