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Autor: Hamenstädt, Ursula
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THEOREM B. *For every $k \geq 2$ and $g = \frac{k}{2}(k+1)$ the Teichmüller space $\mathcal{T}_{g,0}$ can be parametrized by the length functions of $6g+3$ free homotopy classes contained in the orbit of a fixed class under a maximal finite subgroup G of $\text{Map}(g,0)$. The group G is a semidirect product of a cyclic group of order $2g+1$ and a cyclic group of order 3.*

We refer to [S2] for a discussion of other interesting parametrizations of $\mathcal{T}_{g,0}$.

The structure of this note is as follows. In Section 2 we look at simple triangle surfaces with additional symmetries. In Section 3 we give a combinatorial description of a family of curves which contains the systoles of every simple triangle surface. Length estimates in Section 4 lead to a complete description of the systoles of a simple triangle surface. This is used in Section 5 to show our theorems.

As a notational convention, we number the vertices of a fundamental $2p$ -gon Ω counter-clockwise in consecutive order and we number and orient the edges of Ω in such a way that the edge i as an oriented arc joins the vertex $i-1$ to the vertex i . Moreover we write simply \mathcal{T}_g for the Teichmüller space of marked hyperbolic structures on a closed surface of genus g .

2. BASIC PROPERTIES OF SIMPLE TRIANGLE SURFACES

Let $g \geq 2$ and let $p = 2g+1$. There is up to isometry a unique $2p$ -gon Ω in the hyperbolic plane \mathbf{H}^2 with geodesic sides of equal length and with angles $2\pi/p$. In the introduction we called Ω a *fundamental $2p$ -gon*. The *center* of Ω is the unique point $z \in \Omega$ which has the same distance to each of the vertices. A fundamental $2p$ -gon admits a cyclic group Γ of isometries whose elements rotate Ω about the center with a rotation angle which is a multiple of $2\pi/p$. We view Γ as a cyclic group of isometries of the whole hyperbolic plane \mathbf{H}^2 .

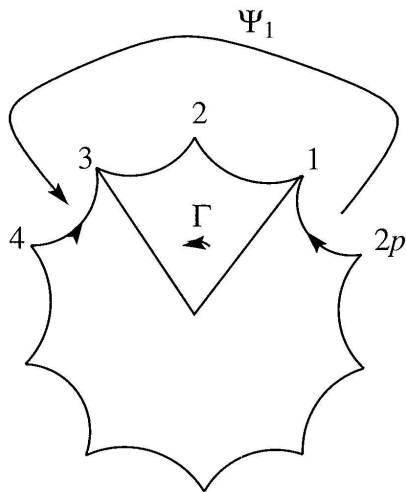
We call a closed hyperbolic surface S of genus g a *simple triangle surface* if $S = \mathbf{H}^2/G$ where G is a discrete torsion free group $G \subset \text{PSL}(2, \mathbf{R})$ of isometries of \mathbf{H}^2 which is normalized by the group Γ and which admits Ω as a fundamental polygon (see [M] for basic informations on fundamental polygons). The group G then acts as a group of side pairing transformations for the polygon Ω . This means that for each side a of Ω there is an isometry $\Psi \in G$ which maps a to a second side $\Psi(a) \neq a$ of Ω in such a way that $\Psi(\Omega) \cap \Omega = \Psi a$.

Our first observation is that simple triangle surfaces exist for every genus $g \geq 2$.

LEMMA 2.1. *For every $g \geq 2$ there is a simple triangle surface of genus g .*

Proof. Let $p \geq 5$ be an odd number and let Ω be a fundamental $2p$ -gon with center $0 \in \mathbf{H}^2$. We have to show that there is a discrete subgroup G of $PSL(2, \mathbf{R})$ which is normalized by Γ and which admits Ω as a fundamental polygon.

Choose a number $k \in \{2, \dots, p-1\}$ and define a family $\{\Psi_1, \dots, \Psi_p\}$ of isometries of \mathbf{H}^2 by requiring that Ψ_j maps the (oriented) edge with odd number $2j+1$ orientation reversing onto the (oriented) edge $2j+2k$ in such a way that $\Psi_j(\Omega) \cap \Omega$ is just the edge $2j+2k$. Then necessarily the vertex $2j$ is mapped to the vertex $2j+2k$, and the vertex $2j+1$ is mapped to the vertex $2j+2k-1$. We claim that these isometries $\{\Psi_1, \dots, \Psi_p\}$ generate a discrete subgroup of $PSL(2, \mathbf{R})$ with fundamental domain Ω if and only if k and $k-1$ are prime to p .



To see this let G be the subgroup of $PSL(2, \mathbf{R})$ generated by Ψ_1, \dots, Ψ_p and assume that G is discrete and torsion free, with fundamental polygon Ω . By the choice of Ψ_1, \dots, Ψ_p , the G -orbit of an even (or odd) vertex of Ω intersects Ω only in the set of even (or odd) vertices. Different such vertex cycles project to different points on the surface $S = \mathbf{H}^2/G$. If $m \geq 2$ is the number of points in the vertex cycle of the vertex a , then a neighborhood of the projection \bar{a} of a to S consists of $2m$ equilateral hyperbolic triangles with angle π/p which contain \bar{a} as one of their vertices. Since S is a smooth hyperbolic surface, the angles at \bar{a} of these triangles must add up to 2π . This means that there are precisely 2 vertex cycles for the action of G , each

containing only even or only odd vertices. By the definition of G this is the case if and only if the number $k \in \{2, \dots, p-1\}$ is prime to p and $k-1$ is prime to p as well. Such a group G is then normalized by the group Γ of rotations of Ω with rotation angle a multiple of 2π .

The same argument also shows that for $k \in \{2, \dots, p-1\}$ which is prime to p and such that $k-1$ is prime to p as well the group G induces a simple triangle surface of genus g . Since $p = 2g + 1$ is odd we can always choose $k = 2$ to obtain an example. \square

In the above proof we observed that we obtain a simple triangle surface from a fundamental $2p$ -gon Ω by identifying the edge 1 with the edge $2k$ for some $k \in \{2, \dots, p-1\}$ if and only if k and $k-1$ are prime to p . We denote by $S(p; k)$ the surface obtained in this way. For fixed $p \geq 5$ this defines a finite non-empty collection of simple triangle surfaces of genus $\frac{1}{2}p - 1$ indexed by the set of all numbers $k \in \{2, \dots, p-1\}$ which are prime to p and such that $k-1$ is prime to p as well. However these surfaces are not necessarily distinct as hyperbolic surfaces. For example, via exchanging the roles of the even and odd vertices of our fundamental $2p$ -gon Ω we observe that the surface $S(p; k)$ is isometric to the surface $S(p; p-k+1)$. Thus we may restrict our attention to the case that $k \leq \frac{1}{2}(p+1)$. In the sequel we sometimes identify the surfaces $S(p; k)$ and $S(p; p-k+1)$ without further comment.

Let again Γ be the group of rotations of Ω which descends to a group of isometries on a simple triangle surface S of genus g . The natural Γ -invariant triangulation of Ω into $2p$ equilateral triangles with angle π/p projects to the Γ -invariant canonical triangulation whose 3 vertices $0, A, B$ are just the fixed points for the action of Γ . The quotient S/Γ of S under Γ is a topological 2-sphere. The hyperbolic metric on S projects to a hyperbolic metric on S/Γ with 3 singular points $\hat{A}, \hat{B}, \hat{0}$ which are the projections of the vertices $A, B, 0$ of the canonical triangulation of S . With this metric, S/Γ is isometric to two equilateral hyperbolic triangles with angle π/p glued at their boundaries. This observation is used in the proof of the following.

LEMMA 2.2.

1) Let $p \geq 5$ be an odd number and let $k, m \in \{2, \dots, p-1\}$ be numbers which are prime to p and such that $k-1, m-1$ are prime to p as well. If either $(k-1)m+1 \equiv 0 \pmod{p}$ or $(m-1)k+1 \equiv 0 \pmod{p}$ then the surfaces $S(p; k)$ and $S(p; m)$ are isometric.

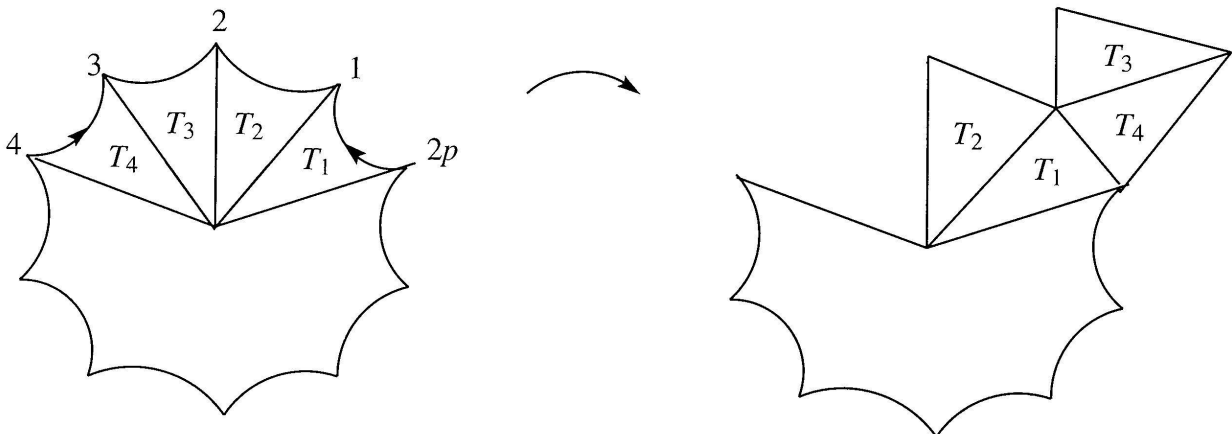
2) A simple triangle surface S with basic group Γ of isometries admits a nontrivial group $\Sigma \not\subset \Gamma$ of orientation preserving isometries which normalizes Γ if and only if one of the following holds.

- i) $S = S(p; k)$ for some $k \geq 2$ and a divisor $p \geq k + 1$ of $k(k - 1) + 1$. The group Σ is then cyclic of order 3.
- ii) $S = S(p; 2)$ and the group Σ is cyclic of order 2 and generated by a hyperelliptic involution.

Proof. Let $p \geq 5$ and let $k \leq p - 1$ be such that $k - 1$ and k are prime to p . Let Ω be a fundamental $2p$ -gon and let $0, A, B$ be the vertices of the canonical triangulation of S . We assume that 0 is the projection of the center of Ω and A is the projection of the odd vertices of the boundary of Ω .

As in the introduction we number the $2p$ edges of Ω in counterclockwise order in such a way that the edge i is adjacent to the vertices $i - 1$ and i . Let $T_i \subset S$ be the projection of the triangle in Ω with one vertex at the center of Ω and with the edge i of Ω as the opposite side. The triangles T_1, \dots, T_{2p} are arranged in counterclockwise order around the vertex 0 .

There is a different representation of S as a quotient of Ω under a group of side pairing transformations in such a way that the center of Ω projects to the vertex A of the canonical triangulation. Namely, if we cut S open along the geodesic arcs connecting the vertices 0 and B , then the result is a fundamental $2p$ -gon which consists again of the triangles T_1, \dots, T_{2p} . The arrangement of these triangles around the vertex A is given by a permutation σ of $\{1, \dots, 2p\}$ with $\sigma(1) = 1$, i.e. the counterclockwise order of the triangles around the vertex A is $T_{\sigma(1)}, \dots, T_{\sigma(2p)}$. The parity of $\sigma(i)$ coincides with the parity of i . Moreover for every $i \in \{1, \dots, p\}$ we have $\sigma(2i) = \sigma(2i + 1) + 1 \pmod{2p}$.



The side pairings of Ω which define S in such a way that the center of Ω projects to 0 glue the edge $2k$ to the edge 1 and therefore we have

$\sigma(2) = 2k$ and $\sigma(3) = 2k - 1$. The basic group Γ of isometries of S permutes the triangles T_i and fixes the vertex A . This implies that σ normalizes the group of permutations of $\{1, \dots, 2p\}$ generated by the permutation $\tau(i) = i + 2 \pmod{2p}$ and hence necessarily $\sigma(2i) = 2i(k - 1) + 2$.

To obtain our surface S we have to identify the edge $2i - 1$ with the edge $2im$ for some $m \in \{2, \dots, p - 1\}$ with an orientation reversing isometry. The number m is uniquely determined if we require in addition that the triangles adjacent to odd edges of Ω project once again to the triangles T_{2i-1} ($i = 1, \dots, p$) of the canonical triangulation.

Comparing the arrangement of triangles around 0 and A we conclude that $\sigma(2m) = 2p$. Together with the above this shows that $2m(k - 1) + 2 \equiv 0 \pmod{2p}$ or, equivalently, $m(k - 1) + 1 \equiv 0 \pmod{p}$. In other words, if $m, k \geq 2$ are such that $m(k - 1) + 1 \equiv 0 \pmod{p}$ then the surfaces $S(p; k)$ and $S(p; m)$ are isometric. This shows the first part of the lemma.

To show the second part of our lemma let S be a simple triangle surface which admits a non-trivial group Σ of orientation preserving isometries normalizing the basic group Γ . Then the action of Σ on S descends to an isometric action on the sphere S/Γ . The sphere S/Γ consists of two equilateral triangles with angle π/p glued at their boundaries. One of these triangles is the projection of the odd triangles of the canonical triangulation of S , the other one is the projection of the even triangles.

Every isometry of S/Γ has to preserve the singular set $\{\hat{A}, \hat{B}, \hat{O}\} \subset S/\Gamma$ of ramification points which consists of the vertices of the two triangles forming S/Γ . The only nontrivial isometry of S/Γ which fixes each of the ramification points $\hat{O}, \hat{A}, \hat{B}$ is the orientation reversing reflection which exchanges the two triangles forming S/Γ . By assumption the elements of Σ preserve the orientation of S and hence of S/Γ , and therefore there are two possibilities:

- 1) Σ contains an element Ψ which permutes cyclicly the singular points $\hat{A}, \hat{B}, \hat{O}$ of S/Γ and preserves each of the two triangles which form S/Γ .
- 2) Σ fixes one singular point of S/Γ , permutes the two other ones and exchanges the two triangles which form S/Γ .

Assume that $S = S(p; k)$ admits an isometry Ψ as in 1) above. Then Ψ permutes the triangles of the canonical triangulation, but preserves their parity. If we cut $S = S(p; k)$ open along those edges of the triangles of the canonical triangulation which connect the vertices A and B , then the result is the fundamental $2p$ -gon Ω and we obtain our surface from Ω by a side pairing which identifies the edges 1 and $2k$. Since Ψ is an isometry of S

which preserves the canonical triangulation, if we cut S open along the edges connecting the vertices $\Psi(A)$ and $\Psi(B)$ then the result is again the polygon Ω , and once again we obtain S from Ω by identifying the edges 1 and $2k$. This together with the above consideration shows that $k(k-1)+1 \equiv 0 \pmod{p}$ and therefore p divides $k(k-1)+1$.

Assume now that S admits an isometry Ψ as in 2) above. Then Ψ permutes the triangles of the canonical triangulation and changes their parity with respect to a given counter clockwise numbering around a given vertex. Let $m \leq p-1$ be such that $k(m-1)+1 \equiv 0 \pmod{p}$. The above considerations imply that necessarily $k = p - m + 1$ and hence $(m-1)^2 \equiv 1 \pmod{p}$ or equivalently $m(m-2) \equiv 0 \pmod{p}$. Since $m \geq 1$ is prime to p we conclude that either $m = 2$ or that p divides $m-2$. But $m \leq p-1$ and therefore only the case $m = 2$ is possible.

We are left with showing that the isometry Ψ is a hyperelliptic involution. For this notice that every fixed point of Ψ projects to a fixed point for the induced isometry $\hat{\Psi}$ of S/Γ . The map $\hat{\Psi}$ has precisely two fixed points: A singular point $\hat{0}$ of S/Γ and the midpoint y of the geodesic arc connecting the two other singular points.

There are exactly $p = 2g + 1$ preimages of y in S . Since $\Psi^2 = Id$ and since Ψ normalizes Γ , either every preimage or no preimage is fixed by Ψ . The Riemann Hurwitz-formula [F] shows that the second case is impossible. Thus Ψ has exactly $p + 1 = 2g + 2$ fixed points and is a hyperelliptic involution. \square

COROLLARY 2.3. *For every $g \geq 2$ there is a hyperelliptic surface of genus g whose full automorphism group is the direct product of a cyclic group of order $2g + 1$ and a cyclic group of order 2 generated by a hyperelliptic involution.*

Proof. We showed in Lemma 2.1 that for each $g \geq 2$ there is a simple triangle surface $S(2g + 1; 2)$. By Lemma 2.2 and its proof, this surface is hyperelliptic and its isometry group is as stated in the corollary. \square

REMARK. There are surfaces $S(p; k)$ for $p \notin \{\ell(\ell-1)+1 \mid \ell \geq 2\}$ which admit a cyclic group Σ of isometries of order 3 contained in the normalizer of the basic group Γ . The simplest surface of this kind is the surface $S(19; 8)$ of genus $g = 9$.