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**THEOREM B.** *For every  $k \geq 2$  and  $g = \frac{k}{2}(k+1)$  the Teichmüller space  $\mathcal{T}_{g,0}$  can be parametrized by the length functions of  $6g+3$  free homotopy classes contained in the orbit of a fixed class under a maximal finite subgroup  $G$  of  $\text{Map}(g,0)$ . The group  $G$  is a semidirect product of a cyclic group of order  $2g+1$  and a cyclic group of order 3.*

We refer to [S2] for a discussion of other interesting parametrizations of  $\mathcal{T}_{g,0}$ .

The structure of this note is as follows. In Section 2 we look at simple triangle surfaces with additional symmetries. In Section 3 we give a combinatorial description of a family of curves which contains the systoles of every simple triangle surface. Length estimates in Section 4 lead to a complete description of the systoles of a simple triangle surface. This is used in Section 5 to show our theorems.

As a notational convention, we number the vertices of a fundamental  $2p$ -gon  $\Omega$  counter-clockwise in consecutive order and we number and orient the edges of  $\Omega$  in such a way that the edge  $i$  as an oriented arc joins the vertex  $i-1$  to the vertex  $i$ . Moreover we write simply  $\mathcal{T}_g$  for the Teichmüller space of marked hyperbolic structures on a closed surface of genus  $g$ .

## 2. BASIC PROPERTIES OF SIMPLE TRIANGLE SURFACES

Let  $g \geq 2$  and let  $p = 2g+1$ . There is up to isometry a unique  $2p$ -gon  $\Omega$  in the hyperbolic plane  $\mathbf{H}^2$  with geodesic sides of equal length and with angles  $2\pi/p$ . In the introduction we called  $\Omega$  a *fundamental  $2p$ -gon*. The *center* of  $\Omega$  is the unique point  $z \in \Omega$  which has the same distance to each of the vertices. A fundamental  $2p$ -gon admits a cyclic group  $\Gamma$  of isometries whose elements rotate  $\Omega$  about the center with a rotation angle which is a multiple of  $2\pi/p$ . We view  $\Gamma$  as a cyclic group of isometries of the whole hyperbolic plane  $\mathbf{H}^2$ .

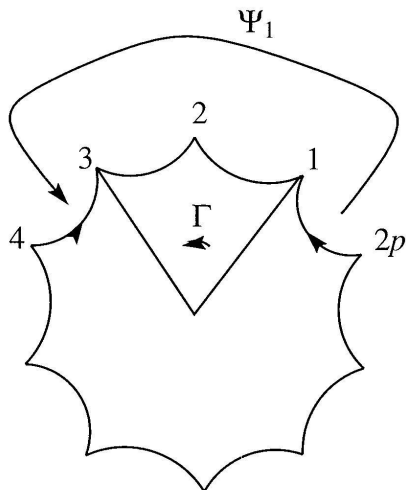
We call a closed hyperbolic surface  $S$  of genus  $g$  a *simple triangle surface* if  $S = \mathbf{H}^2/G$  where  $G$  is a discrete torsion free group  $G \subset \text{PSL}(2, \mathbf{R})$  of isometries of  $\mathbf{H}^2$  which is normalized by the group  $\Gamma$  and which admits  $\Omega$  as a fundamental polygon (see [M] for basic informations on fundamental polygons). The group  $G$  then acts as a group of side pairing transformations for the polygon  $\Omega$ . This means that for each side  $a$  of  $\Omega$  there is an isometry  $\Psi \in G$  which maps  $a$  to a second side  $\Psi(a) \neq a$  of  $\Omega$  in such a way that  $\Psi(\Omega) \cap \Omega = \Psi a$ .

Our first observation is that simple triangle surfaces exist for every genus  $g \geq 2$ .

LEMMA 2.1. *For every  $g \geq 2$  there is a simple triangle surface of genus  $g$ .*

*Proof.* Let  $p \geq 5$  be an odd number and let  $\Omega$  be a fundamental  $2p$ -gon with center  $0 \in \mathbf{H}^2$ . We have to show that there is a discrete subgroup  $G$  of  $PSL(2, \mathbf{R})$  which is normalized by  $\Gamma$  and which admits  $\Omega$  as a fundamental polygon.

Choose a number  $k \in \{2, \dots, p-1\}$  and define a family  $\{\Psi_1, \dots, \Psi_p\}$  of isometries of  $\mathbf{H}^2$  by requiring that  $\Psi_j$  maps the (oriented) edge with odd number  $2j+1$  orientation reversing onto the (oriented) edge  $2j+2k$  in such a way that  $\Psi_j(\Omega) \cap \Omega$  is just the edge  $2j+2k$ . Then necessarily the vertex  $2j$  is mapped to the vertex  $2j+2k$ , and the vertex  $2j+1$  is mapped to the vertex  $2j+2k-1$ . We claim that these isometries  $\{\Psi_1, \dots, \Psi_p\}$  generate a discrete subgroup of  $PSL(2, \mathbf{R})$  with fundamental domain  $\Omega$  if and only if  $k$  and  $k-1$  are prime to  $p$ .



To see this let  $G$  be the subgroup of  $PSL(2, \mathbf{R})$  generated by  $\Psi_1, \dots, \Psi_p$  and assume that  $G$  is discrete and torsion free, with fundamental polygon  $\Omega$ . By the choice of  $\Psi_1, \dots, \Psi_p$ , the  $G$ -orbit of an even (or odd) vertex of  $\Omega$  intersects  $\Omega$  only in the set of even (or odd) vertices. Different such vertex cycles project to different points on the surface  $S = \mathbf{H}^2/G$ . If  $m \geq 2$  is the number of points in the vertex cycle of the vertex  $a$ , then a neighborhood of the projection  $\bar{a}$  of  $a$  to  $S$  consists of  $2m$  equilateral hyperbolic triangles with angle  $\pi/p$  which contain  $\bar{a}$  as one of their vertices. Since  $S$  is a smooth hyperbolic surface, the angles at  $\bar{a}$  of these triangles must add up to  $2\pi$ . This means that there are precisely 2 vertex cycles for the action of  $G$ , each

containing only even or only odd vertices. By the definition of  $G$  this is the case if and only if the number  $k \in \{2, \dots, p-1\}$  is prime to  $p$  and  $k-1$  is prime to  $p$  as well. Such a group  $G$  is then normalized by the group  $\Gamma$  of rotations of  $\Omega$  with rotation angle a multiple of  $2\pi$ .

The same argument also shows that for  $k \in \{2, \dots, p-1\}$  which is prime to  $p$  and such that  $k-1$  is prime to  $p$  as well the group  $G$  induces a simple triangle surface of genus  $g$ . Since  $p = 2g + 1$  is odd we can always choose  $k = 2$  to obtain an example.  $\square$

In the above proof we observed that we obtain a simple triangle surface from a fundamental  $2p$ -gon  $\Omega$  by identifying the edge 1 with the edge  $2k$  for some  $k \in \{2, \dots, p-1\}$  if and only if  $k$  and  $k-1$  are prime to  $p$ . We denote by  $S(p; k)$  the surface obtained in this way. For fixed  $p \geq 5$  this defines a finite non-empty collection of simple triangle surfaces of genus  $\frac{1}{2}p - 1$  indexed by the set of all numbers  $k \in \{2, \dots, p-1\}$  which are prime to  $p$  and such that  $k-1$  is prime to  $p$  as well. However these surfaces are not necessarily distinct as hyperbolic surfaces. For example, via exchanging the roles of the even and odd vertices of our fundamental  $2p$ -gon  $\Omega$  we observe that the surface  $S(p; k)$  is isometric to the surface  $S(p; p-k+1)$ . Thus we may restrict our attention to the case that  $k \leq \frac{1}{2}(p+1)$ . In the sequel we sometimes identify the surfaces  $S(p; k)$  and  $S(p; p-k+1)$  without further comment.

Let again  $\Gamma$  be the group of rotations of  $\Omega$  which descends to a group of isometries on a simple triangle surface  $S$  of genus  $g$ . The natural  $\Gamma$ -invariant triangulation of  $\Omega$  into  $2p$  equilateral triangles with angle  $\pi/p$  projects to the  $\Gamma$ -invariant canonical triangulation whose 3 vertices  $0, A, B$  are just the fixed points for the action of  $\Gamma$ . The quotient  $S/\Gamma$  of  $S$  under  $\Gamma$  is a topological 2-sphere. The hyperbolic metric on  $S$  projects to a hyperbolic metric on  $S/\Gamma$  with 3 singular points  $\hat{A}, \hat{B}, \hat{0}$  which are the projections of the vertices  $A, B, 0$  of the canonical triangulation of  $S$ . With this metric,  $S/\Gamma$  is isometric to two equilateral hyperbolic triangles with angle  $\pi/p$  glued at their boundaries. This observation is used in the proof of the following.

#### LEMMA 2.2.

1) Let  $p \geq 5$  be an odd number and let  $k, m \in \{2, \dots, p-1\}$  be numbers which are prime to  $p$  and such that  $k-1, m-1$  are prime to  $p$  as well. If either  $(k-1)m+1 \equiv 0 \pmod{p}$  or  $(m-1)k+1 \equiv 0 \pmod{p}$  then the surfaces  $S(p; k)$  and  $S(p; m)$  are isometric.

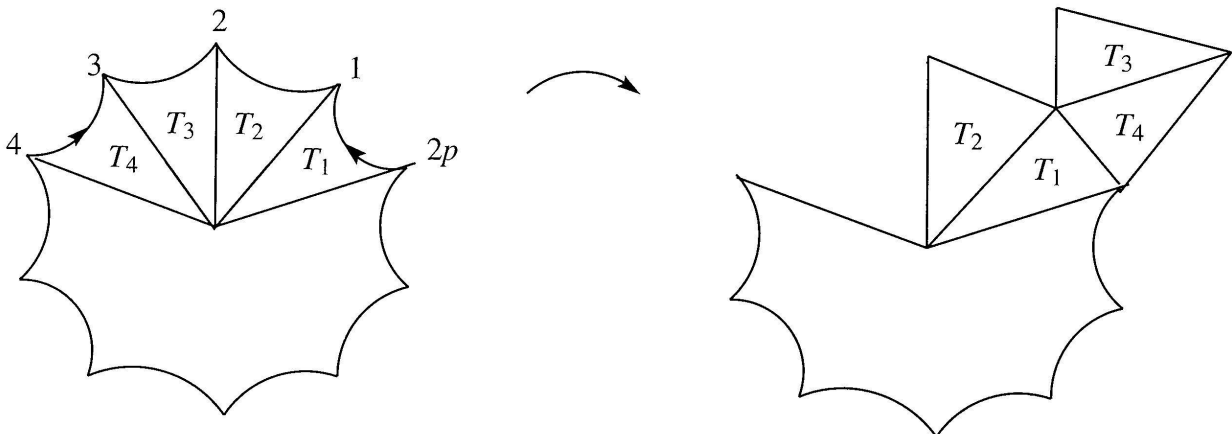
2) A simple triangle surface  $S$  with basic group  $\Gamma$  of isometries admits a nontrivial group  $\Sigma \not\subset \Gamma$  of orientation preserving isometries which normalizes  $\Gamma$  if and only if one of the following holds.

- i)  $S = S(p; k)$  for some  $k \geq 2$  and a divisor  $p \geq k + 1$  of  $k(k - 1) + 1$ . The group  $\Sigma$  is then cyclic of order 3.
- ii)  $S = S(p; 2)$  and the group  $\Sigma$  is cyclic of order 2 and generated by a hyperelliptic involution.

*Proof.* Let  $p \geq 5$  and let  $k \leq p - 1$  be such that  $k - 1$  and  $k$  are prime to  $p$ . Let  $\Omega$  be a fundamental  $2p$ -gon and let  $0, A, B$  be the vertices of the canonical triangulation of  $S$ . We assume that  $0$  is the projection of the center of  $\Omega$  and  $A$  is the projection of the odd vertices of the boundary of  $\Omega$ .

As in the introduction we number the  $2p$  edges of  $\Omega$  in counterclockwise order in such a way that the edge  $i$  is adjacent to the vertices  $i - 1$  and  $i$ . Let  $T_i \subset S$  be the projection of the triangle in  $\Omega$  with one vertex at the center of  $\Omega$  and with the edge  $i$  of  $\Omega$  as the opposite side. The triangles  $T_1, \dots, T_{2p}$  are arranged in counterclockwise order around the vertex  $0$ .

There is a different representation of  $S$  as a quotient of  $\Omega$  under a group of side pairing transformations in such a way that the center of  $\Omega$  projects to the vertex  $A$  of the canonical triangulation. Namely, if we cut  $S$  open along the geodesic arcs connecting the vertices  $0$  and  $B$ , then the result is a fundamental  $2p$ -gon which consists again of the triangles  $T_1, \dots, T_{2p}$ . The arrangement of these triangles around the vertex  $A$  is given by a permutation  $\sigma$  of  $\{1, \dots, 2p\}$  with  $\sigma(1) = 1$ , i.e. the counterclockwise order of the triangles around the vertex  $A$  is  $T_{\sigma(1)}, \dots, T_{\sigma(2p)}$ . The parity of  $\sigma(i)$  coincides with the parity of  $i$ . Moreover for every  $i \in \{1, \dots, p\}$  we have  $\sigma(2i) = \sigma(2i + 1) + 1 \pmod{2p}$ .



The side pairings of  $\Omega$  which define  $S$  in such a way that the center of  $\Omega$  projects to  $0$  glue the edge  $2k$  to the edge  $1$  and therefore we have

$\sigma(2) = 2k$  and  $\sigma(3) = 2k - 1$ . The basic group  $\Gamma$  of isometries of  $S$  permutes the triangles  $T_i$  and fixes the vertex  $A$ . This implies that  $\sigma$  normalizes the group of permutations of  $\{1, \dots, 2p\}$  generated by the permutation  $\tau(i) = i + 2 \pmod{2p}$  and hence necessarily  $\sigma(2i) = 2i(k - 1) + 2$ .

To obtain our surface  $S$  we have to identify the edge  $2i - 1$  with the edge  $2im$  for some  $m \in \{2, \dots, p - 1\}$  with an orientation reversing isometry. The number  $m$  is uniquely determined if we require in addition that the triangles adjacent to odd edges of  $\Omega$  project once again to the triangles  $T_{2i-1}$  ( $i = 1, \dots, p$ ) of the canonical triangulation.

Comparing the arrangement of triangles around 0 and  $A$  we conclude that  $\sigma(2m) = 2p$ . Together with the above this shows that  $2m(k - 1) + 2 \equiv 0 \pmod{2p}$  or, equivalently,  $m(k - 1) + 1 \equiv 0 \pmod{p}$ . In other words, if  $m, k \geq 2$  are such that  $m(k - 1) + 1 \equiv 0 \pmod{p}$  then the surfaces  $S(p; k)$  and  $S(p; m)$  are isometric. This shows the first part of the lemma.

To show the second part of our lemma let  $S$  be a simple triangle surface which admits a non-trivial group  $\Sigma$  of orientation preserving isometries normalizing the basic group  $\Gamma$ . Then the action of  $\Sigma$  on  $S$  descends to an isometric action on the sphere  $S/\Gamma$ . The sphere  $S/\Gamma$  consists of two equilateral triangles with angle  $\pi/p$  glued at their boundaries. One of these triangles is the projection of the odd triangles of the canonical triangulation of  $S$ , the other one is the projection of the even triangles.

Every isometry of  $S/\Gamma$  has to preserve the singular set  $\{\hat{A}, \hat{B}, \hat{O}\} \subset S/\Gamma$  of ramification points which consists of the vertices of the two triangles forming  $S/\Gamma$ . The only nontrivial isometry of  $S/\Gamma$  which fixes each of the ramification points  $\hat{O}, \hat{A}, \hat{B}$  is the orientation reversing reflection which exchanges the two triangles forming  $S/\Gamma$ . By assumption the elements of  $\Sigma$  preserve the orientation of  $S$  and hence of  $S/\Gamma$ , and therefore there are two possibilities:

- 1)  $\Sigma$  contains an element  $\Psi$  which permutes cyclicly the singular points  $\hat{A}, \hat{B}, \hat{O}$  of  $S/\Gamma$  and preserves each of the two triangles which form  $S/\Gamma$ .
- 2)  $\Sigma$  fixes one singular point of  $S/\Gamma$ , permutes the two other ones and exchanges the two triangles which form  $S/\Gamma$ .

Assume that  $S = S(p; k)$  admits an isometry  $\Psi$  as in 1) above. Then  $\Psi$  permutes the triangles of the canonical triangulation, but preserves their parity. If we cut  $S = S(p; k)$  open along those edges of the triangles of the canonical triangulation which connect the vertices  $A$  and  $B$ , then the result is the fundamental  $2p$ -gon  $\Omega$  and we obtain our surface from  $\Omega$  by a side pairing which identifies the edges 1 and  $2k$ . Since  $\Psi$  is an isometry of  $S$

which preserves the canonical triangulation, if we cut  $S$  open along the edges connecting the vertices  $\Psi(A)$  and  $\Psi(B)$  then the result is again the polygon  $\Omega$ , and once again we obtain  $S$  from  $\Omega$  by identifying the edges 1 and  $2k$ . This together with the above consideration shows that  $k(k-1)+1 \equiv 0 \pmod{p}$  and therefore  $p$  divides  $k(k-1)+1$ .

Assume now that  $S$  admits an isometry  $\Psi$  as in 2) above. Then  $\Psi$  permutes the triangles of the canonical triangulation and changes their parity with respect to a given counter clockwise numbering around a given vertex. Let  $m \leq p-1$  be such that  $k(m-1)+1 \equiv 0 \pmod{p}$ . The above considerations imply that necessarily  $k = p - m + 1$  and hence  $(m-1)^2 \equiv 1 \pmod{p}$  or equivalently  $m(m-2) \equiv 0 \pmod{p}$ . Since  $m \geq 1$  is prime to  $p$  we conclude that either  $m = 2$  or that  $p$  divides  $m-2$ . But  $m \leq p-1$  and therefore only the case  $m = 2$  is possible.

We are left with showing that the isometry  $\Psi$  is a hyperelliptic involution. For this notice that every fixed point of  $\Psi$  projects to a fixed point for the induced isometry  $\hat{\Psi}$  of  $S/\Gamma$ . The map  $\hat{\Psi}$  has precisely two fixed points: A singular point  $\hat{0}$  of  $S/\Gamma$  and the midpoint  $y$  of the geodesic arc connecting the two other singular points.

There are exactly  $p = 2g + 1$  preimages of  $y$  in  $S$ . Since  $\Psi^2 = Id$  and since  $\Psi$  normalizes  $\Gamma$ , either every preimage or no preimage is fixed by  $\Psi$ . The Riemann Hurwitz-formula [F] shows that the second case is impossible. Thus  $\Psi$  has exactly  $p + 1 = 2g + 2$  fixed points and is a hyperelliptic involution.  $\square$

**COROLLARY 2.3.** *For every  $g \geq 2$  there is a hyperelliptic surface of genus  $g$  whose full automorphism group is the direct product of a cyclic group of order  $2g + 1$  and a cyclic group of order 2 generated by a hyperelliptic involution.*

*Proof.* We showed in Lemma 2.1 that for each  $g \geq 2$  there is a simple triangle surface  $S(2g + 1; 2)$ . By Lemma 2.2 and its proof, this surface is hyperelliptic and its isometry group is as stated in the corollary.  $\square$

**REMARK.** There are surfaces  $S(p; k)$  for  $p \notin \{\ell(\ell-1)+1 \mid \ell \geq 2\}$  which admit a cyclic group  $\Sigma$  of isometries of order 3 contained in the normalizer of the basic group  $\Gamma$ . The simplest surface of this kind is the surface  $S(19; 8)$  of genus  $g = 9$ .