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NEW EXAMPLES OF MAXIMAL SURFACES

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ABSTRACT. We describe all closed hyperbolic triangle surfaces of a particularly simple type which are maximal, i.e. for which the length of the systole is a local maximum in Teichmüller space. We show that this class of triangle surfaces contains exactly three maximal surfaces. One of these surfaces is the well known Klein surface, the other two examples are new.

1. INTRODUCTION

A *Riemann surface of finite type* is a closed Riemann surface from which a finite number $m \geq 0$ of points, the so-called *punctures*, have been deleted. Closed Riemann surfaces (with no punctures) are topologically determined by their genus. In this note we only consider surfaces of genus $g \geq 2$ with $m \geq 0$ punctures. Such a surface admits a family of complete hyperbolic metrics of finite volume. Each of these metrics corresponds to precisely one complex structure of finite type.

The easiest way to describe all such hyperbolic metrics is to look at the *Teichmüller space* $\mathcal{T}_{g,m}$ of *marked* hyperbolic metrics of finite volume on a surface S_0 of genus g with m punctures. This Teichmüller space is the set of all pairs (f,h) where h is a hyperbolic metric on a surface S and f is the homotopy class of a homeomorphism $F\colon S_0 \to S$ of S_0 onto S. The *mapping*

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class group $Map(g,m)$ is the group of all isotopy classes of homeomorphisms of S_0 onto itself. It acts on $\mathcal{T}_{g,m}$ via $\Psi(f,h) = (f \circ \Psi^{-1}, h)$ and identifies those points in $\mathcal{T}_{g,m}$ which correspond to isometric surfaces. With respect to a natural topology, the Teichmüller space $\mathcal{T}_{g,m}$ is homeomorphic to a cell of dimension $6g - 6 + 2m$, and the mapping class group $Map(g,m)$ acts properly discontinuously, but not fixed-point free. The quotient of $\mathcal{T}_{g,m}$ under $Map(g,m)$ is the moduli space of hyperbolic metrics on our surface of genus g with m punctures.

A *systole* of an oriented hyperbolic surface S of finite volume is a simple closed geodesic on S of minimal length. The length of the systole depends on the choice of the hyperbolic metric and defines a $Map(g,m)$-invariant continuous function on $\mathcal{T}_{g,m}$. This function is bounded from above on $\mathcal{T}_{g,m}$ by a constant depending on g and m [Bu] (which however tends to infinity as g tends to infinity [BS]), but it is not bounded from below on $\mathcal{T}_{g,m}$. We refer to [S3] for other interesting properties of this function.

Following Schmutz [S1] we call a point in $\mathcal{T}_{g,m}$ a *maximal surface* if the length of the systole has a local maximum at that point. Maximal surfaces always exist, and Schmutz found in [S1] explicit examples.

The goal of this paper is to look for maximal surfaces among all hyperbolic surfaces which admit a particularly simple combinatorial description. For this recall that every *closed* hyperbolic surface S is given by a discrete torsion free subgroup G of the isometry group $PSL(2,\mathbf{R})$ of the hyperbolic plane $\mathbf{H}^2$ which acts cocompactly on $\mathbf{H}^2$. The surface S then simply equals $\mathbf{H}^2/G$. The *Dirichlet fundamental polygon* for G based at a point $y \in \mathbf{H}^2$ is the set D of all points $z \in \mathbf{H}^2$ with the property that $\operatorname{dist}(z,y) \le \operatorname{dist}(z,\Psi y)$ for every $\Psi \in G$, where dist is the distance function of the hyperbolic metric. This set is a convex hyperbolic polygon.

For a number $p \ge 5$ define a *fundamental* $2p$*-gon* to be a regular $2p$-gon Ω in the hyperbolic plane $\mathbf{H}^2$ with angles $2\pi/p$ and sides of equal length. Such a $2p$-gon admits a cyclic group Γ of order p of isometries whose elements rotate Ω about a fixed point, with a multiple of $2\pi/p$ as rotation angle. We call the fixed point of the elements of Γ the *center* of Ω. If we draw $2p$ geodesic segments from the center 0 to the vertices of the boundary $\partial\Omega$ of Ω, then these segments decompose Ω into $2p$ equilateral triangles with angle π/p.

We call a closed surface $S = \mathbf{H}^2/G$ a *simple triangle surface* if G admits a fundamental $2p$-gon Ω as the Dirichlet fundamental polygon based at the center of Ω and if moreover G is normalized by the cyclic group Γ. The action of Γ on $\mathbf{H}^2$ then descends to an isometric action on $S = \mathbf{H}^2/G$. We call this group of isometries the *basic group of isometries* of S. The Gauß-Bonnet formula shows that the genus g of S equals $\frac{1}{2}(p-1)$. In particular, the number p is odd.

We number the vertices of $\partial\Omega$ in counter-clockwise order. These vertices are contained in exactly two vertex cycles under the action of Γ. One of these vertex cycles contains the even vertices, the other contains the odd vertices. The triangulation of Ω into $2p$ equilateral triangles with vertices at 0 and on the boundary $\partial\Omega$ of Ω descends to a triangulation of the quotient surface S with 3 vertices. We call this triangulation the *canonical triangulation* of S. If we delete the vertices of the canonical triangulation from the surface S then we obtain a surface of genus g with 3 punctures together with a complex structure of finite type which is invariant under the natural action of the basic group of isometries of S. The unique hyperbolic metric of finite volume which defines this complex structure is again invariant under this group of isometries. In other words, to every simple triangle surface S of genus g corresponds a point S_∞ in the Teichmüller space $\mathcal{T}_{g,3}$ of surfaces of genus g with 3 punctures which we call the *ideal surface* S_∞ associated to S.

The main purpose of this note is to show.

THEOREM A.

1) *Among the simple triangle surfaces there are exactly* 3 *which are maximal. They are of genus* 3, 6 *and* 10.

2) *The ideal surface associated to a simple triangle surface* S *is maximal if and only if* S *is maximal.*

The maximal surface of genus 3 listed in the above theorem is the well known Klein's surface of genus 3 and appears already in the list of maximal surfaces given by Schmutz in [S1] (compare also the proceedings volume [L] about Klein's surface). The examples of genus 6 and genus 10 are new. We remark that by construction our simple triangle surfaces are indeed triangle surfaces in the usual sense, i.e. their isometry group is a nontrivial finite quotient of a triangle group.

From the proof of Theorem A we obtain additional informations on some of the Teichmüller spaces $\mathcal{T}_{g,0}$. To explain this let $[\gamma]$ be a nontrivial free homotopy class on the closed base surface S_0 of genus g. For every point $(f,h) \in \mathcal{T}_{g,0}$, the class $f[\gamma]$ can be represented by a unique closed geodesic with respect to the hyperbolic metric h. The length of this geodesic defines a continuous function on $\mathcal{T}_{g,0}$. We call this function the *length function* of $[\gamma]$. We show

THEOREM B. *For every* $k \geq 2$ *and* $g = \frac{k}{2}(k+1)$ *the Teichmüller space* $\mathcal{T}_{g,0}$ *can be parametrized by the length functions of* $6g+3$ *free homotopy classes contained in the orbit of a fixed class under a maximal finite subgroup* G *of* $Map(g,0)$. *The group* G *is a semidirect product of a cyclic group of order* $2g+1$ *and a cyclic group of order* 3.

We refer to [S2] for a discussion of other interesting parametrizations of $\mathcal{T}_{g,0}$.

The structure of this note is as follows. In Section 2 we look at simple triangle surfaces with additional symmetries. In Section 3 we give a combinatorial description of a family of curves which contains the systoles of every simple triangle surface. Length estimates in Section 4 lead to a complete description of the systoles of a simple triangle surface. This is used in Section 5 to show our theorems.

As a notational convention, we number the vertices of a fundamental $2p$-gon Ω counter-clockwise in consecutive order and we number and orient the edges of Ω in such a way that the edge i as an oriented arc joins the vertex $i-1$ to the vertex i. Moreover we write simply $\mathcal{T}_g$ for the Teichmüller space of marked hyperbolic structures on a closed surface of genus g.

2. BASIC PROPERTIES OF SIMPLE TRIANGLE SURFACES

Let $g \geq 2$ and let $p = 2g+1$. There is up to isometry a unique $2p$-gon Ω in the hyperbolic plane $\mathbf{H}^2$ with geodesic sides of equal length and with angles $2\pi/p$. In the introduction we called Ω a *fundamental* $2p$*-gon.* The *center* of Ω is the unique point $z \in \Omega$ which has the same distance to each of the vertices. A fundamental $2p$-gon admits a cyclic group Γ of isometries whose elements rotate Ω about the center with a rotation angle which is a multiple of $2\pi/p$. We view Γ as a cyclic group of isometries of the whole hyperbolic plane $\mathbf{H}^2$.

We call a closed hyperbolic surface S of genus g a *simple triangle surface* if $S = \mathbf{H}^2/G$ where G is a discrete torsion free group $G \subset PSL(2,\mathbf{R})$ of isometries of $\mathbf{H}^2$ which is normalized by the group Γ and which admits Ω as a fundamental polygon (see [M] for basic informations on fundamental polygons). The group G then acts as a group of side pairing transformations for the polygon Ω. This means that for each side a of Ω there is an isometry $\Psi \in G$ which maps a to a second side $\Psi(a) \neq a$ of Ω in such a way that $\Psi(\Omega) \cap \Omega = \Psi a$.