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## 5. CONCLUDING COMMENTS

In order to solve the local equivalence problem (i.e. when two metrics  $\mathbf{g}_1, \mathbf{g}_2$  on a differentiable manifold  $M^n$  differ (locally) by a diffeomorphism), Riemann tried to compute  $n\frac{n-1}{2}$   $\text{Diff}(M^n)$ -equivariant functions (i.e.  $K(\mathbf{g}_2)(p) = K(\mathbf{g}_1)(f(p))$  for all  $f \in \text{Diff}(M^n)$ ,  $p \in M^n$ ,  $\mathbf{g}_2 = f^*\mathbf{g}_1$ ). The Gaussian curvature  $K$  is such a function when  $n = 2$ . To do this, Riemann expanded the metric in normal coordinates and defined a map  $Q$  from  $\mathcal{M}_n$ , the space of Riemannian metrics on  $M^n$ , to  $C^\infty(G_2(M^n))$ , where  $G_2(M^n)$  is the two-Grassmannian bundle over  $M^n$ . In other words,  $Q(\mathbf{g})(\pi_p)$  is the sectional curvature of the 2-plane  $\pi_p$  at  $p \in M^n$  with respect to the metric  $\mathbf{g}$ . Then he said that "... if the curvature is given in  $n\frac{n-1}{2}$  surface directions at every point, then the metric relations of the manifold may be determined ..." [Sp2, p. 144]. More precisely, Riemann took  $n\frac{n-1}{2}$  independent sections  $\pi_{ij}$  of the bundle  $G_2(M^n)$  and he defined the  $n\frac{n-1}{2}$  functions by composing with  $Q$  (i.e. a map from  $\mathcal{M}_n$  to  $\{C^\infty(M^n)\}^{n\frac{n-1}{2}}$ ). Perhaps the expression of  $Q$  in coordinates, the two-dimensional flat case and the counting argument led Riemann to the wrong conclusion that  $Q$  can be recovered from evaluation in  $n\frac{n-1}{2}$  independent 2-planes. It is hard to believe that he did not observe that this map is not actually a  $\text{Diff}(M^n)$ -equivariant morphism, as follows from the fact that a generic diffeomorphism does not preserve the  $\pi_{ij}$  (i.e.  $f^*\pi_{ij} \neq \pi_{ij}$ ) when  $n > 2$ .

REMARK 5.1. A way of defining  $n\frac{n-1}{2}$   $\text{Diff}(M^n)$ -equivariant functions from  $\mathcal{M}_n$  to  $C^\infty(M^n)$  such that:

(i) if  $n = 2$  then the function is the Gauss curvature  $K$ ;

(ii) if the  $n\frac{n-1}{2}$  functions vanish identically then the metric  $\mathbf{g}$  is flat;

is as follows. Regarding the curvature tensor  $R$  as a symmetric endomorphism of the second exterior product bundle  $\bigwedge^2(M^n)$  one can take the characteristic polynomial  $\chi_R(X)$  of  $R$ . Then the coefficients of  $\chi_R(X)$  are the required  $n\frac{n-1}{2}$  functions.

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