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Autor: Di SCALA, Antonio J.
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5. Concluding comments

In order to solve the local equivalence problem (i.e. when two metrics $\mathbf{g}_1, \mathbf{g}_2$ on a differentiable manifold M^n differ (locally) by a diffeomorphism), Riemann tried to compute $n\frac{n-1}{2}$ $\mathrm{Diff}(M^n)$-equivariant functions (i.e. $K(\mathbf{g}_2)(p) = K(\mathbf{g}_1)(f(p))$ for all $f \in \mathrm{Diff}(M^n)$, $p \in M^n$, $\mathbf{g}_2 = f^*\mathbf{g}_1$). The Gaussian curvature K is such a function when $n = 2$. To do this, Riemann expanded the metric in normal coordinates and defined a map Q from $\mathcal{M}_n$, the space of Riemannian metrics on M^n, to $C^\infty(G_2(M^n))$, where $G_2(M^n)$ is the two-Grassmannian bundle over M^n. In other words, $Q(\mathbf{g})(\pi_p)$ is the sectional curvature of the 2-plane π_p at $p \in M^n$ with respect to the metric $\mathbf{g}$. Then he said that "... if the curvature is given in $n\frac{n-1}{2}$ surface directions at every point, then the metric relations of the manifold may be determined..." [Sp2, p. 144]. More precisely, Riemann took $n\frac{n-1}{2}$ independent sections π_{ij} of the bundle $G_2(M^n)$ and he defined the $n\frac{n-1}{2}$ functions by composing with Q (i.e. a map from $\mathcal{M}_n$ to $\{C^\infty(M^n)\}^{n\frac{n-1}{2}}$). Perhaps the expression of Q in coordinates, the two-dimensional flat case and the counting argument led Riemann to the wrong conclusion that Q can be recovered from evaluation in $n\frac{n-1}{2}$ independent 2-planes. It is hard to believe that he did not observe that this map is not actually a $\mathrm{Diff}(M^n)$-equivariant morphism, as follows from the fact that a generic diffeomorphism does not preserve the π_{ij} (i.e. $f^*\pi_{ij} \neq \pi_{ij}$) when $n > 2$.

Remark 5.1. A way of defining $n\frac{n-1}{2}$ $\mathrm{Diff}(M^n)$-equivariant functions from $\mathcal{M}_n$ to $C^\infty(M^n)$ such that:

(i) if $n = 2$ then the function is the Gauss curvature K;

(ii) if the $n\frac{n-1}{2}$ functions vanish identically then the metric $\mathbf{g}$ is flat;

is as follows. Regarding the curvature tensor R as a symmetric endomorphism of the second exterior product bundle $\bigwedge^2(M^n)$ one can take the characteristic polynomial $\chi_R(X)$ of R. Then the coefficients of $\chi_R(X)$ are the required $n\frac{n-1}{2}$ functions.

References

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