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PROPOSITION 3.2. *Let* $M(2,b) := S^2 \times H^2 \times T^b$. *There exist local coordinates* $(u_1,\ldots,u_{4+b})$ *on* $M(2,b)$ *such that* $Q(\partial_i^u \wedge \partial_j^u) = 0$ *for* $1 \leq i < j \leq 4+b$.

Let ω be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

LEMMA 3.3. *Let* M^n *be an orientable Riemannian manifold. Then around each point there exists a coordinate system* $\{x_1,\ldots,x_n\}$ *such that* $\omega(\partial_1^x,\ldots,\partial_n^x) = 1$.

Proof of Proposition 3.2. We use Lemma 3.3 to find local coordinates (x_1,x_2) and (y_1,y_2) on S^2 and H^2 such that $\omega(\partial_1^x,\partial_2^x) = 1$ and $\omega(\partial_1^y,\partial_2^y) = 1$. Let $(z_1,\ldots,z_b)$ be the usual flat coordinates on T^b. Define local coordinates on $S^2 \times H^2 \times T^b$ by:

$$u_1 := x_1 + y_1\,, \qquad u_2 := x_1 - y_1\,, \qquad u_3 := x_2 + y_2\,, \qquad u_4 := x_2 - y_2\,,$$

and $u_{k+4} = w_k$ for $1 \leq k \leq b$. We then have

$$\partial_1^u = \partial_1^x + \partial_1^y\,, \qquad \partial_2^u = \partial_1^x - \partial_1^y\,, \qquad \partial_3^u = \partial_2^x + \partial_2^y\,, \qquad \partial_4^u = \partial_2^x + \partial_2^y,$$

and $\partial_{4+k}^u = \partial_k^w$ for $k > 0$. If N is a Riemann surface with constant sectional curvature ϵ, then $\langle R(x,y)y,x\rangle = \epsilon\,\omega(x,y)$. Thus, the calculations performed in the proof of Proposition 3.1 show that $Q(\partial_i^u \wedge \partial_i^v) = 0$. □

4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

PROPOSITION 4.1. *Let* $M = B \times_f F$ *be a warped product, where* B *is a small open ball around* $(0,0)$ *in* $\mathbf{R}^2$, *where* $f(x,y) = x + y + xy + 1$ *is positive, and where* $F = \mathbf{R}$. *Then* M *is not flat. Furthermore* $Q(\partial_x \wedge \partial_y) = 0$, $Q(\partial_x \wedge \partial_z) = 0$, *and* $Q(\partial_y \wedge \partial_z) = 0$.

Proof. We use [O, p. 210, Proposition 42], to compute:

$$\langle R(\partial_x,\partial_y)\,\partial_x,\partial_z\rangle = 0, \qquad \langle R(\partial_x,\partial_z)\,\partial_x,\partial_z\rangle = 0,$$
$$\langle R(\partial_y,\partial_z)\,\partial_y,\partial_z\rangle = 0, \qquad \langle R(\partial_x,\partial_z)\,\partial_z,\partial_y\rangle = f. \qquad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.