

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 47 (2001)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON AN ASSERTION IN RIEMANN'S HABILITATIONSVORTRAG  
**Autor:** Di SCALA, Antonio J.  
**Kapitel:** 4. Curvature zero 2-planes in warped products  
**DOI:** <https://doi.org/10.5169/seals-65428>

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PROPOSITION 3.2. *Let  $M(2, b) := S^2 \times H^2 \times T^b$ . There exist local coordinates  $(u_1, \dots, u_{4+b})$  on  $M(2, b)$  such that  $Q(\partial_i^u \wedge \partial_j^u) = 0$  for  $1 \leq i < j \leq 4 + b$ .*

Let  $\omega$  be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

LEMMA 3.3. *Let  $M^n$  be an orientable Riemannian manifold. Then around each point there exists a coordinate system  $\{x_1, \dots, x_n\}$  such that  $\omega(\partial_1^x, \dots, \partial_n^x) = 1$ .*

*Proof of Proposition 3.2.* We use Lemma 3.3 to find local coordinates  $(x_1, x_2)$  and  $(y_1, y_2)$  on  $S^2$  and  $H^2$  such that  $\omega(\partial_1^x, \partial_2^x) = 1$  and  $\omega(\partial_1^y, \partial_2^y) = 1$ . Let  $(z_1, \dots, z_b)$  be the usual flat coordinates on  $T^b$ . Define local coordinates on  $S^2 \times H^2 \times T^b$  by:

$$u_1 := x_1 + y_1, \quad u_2 := x_1 - y_1, \quad u_3 := x_2 + y_2, \quad u_4 := x_2 - y_2,$$

and  $u_{k+4} = z_k$  for  $1 \leq k \leq b$ . We then have

$$\partial_1^u = \partial_1^x + \partial_1^y, \quad \partial_2^u = \partial_1^x - \partial_1^y, \quad \partial_3^u = \partial_2^x + \partial_2^y, \quad \partial_4^u = \partial_2^x - \partial_2^y,$$

and  $\partial_{4+k}^u = \partial_k^z$  for  $k > 0$ . If  $N$  is a Riemann surface with constant sectional curvature  $\epsilon$ , then  $\langle R(x, y)y, x \rangle = \epsilon \omega(x, y)$ . Thus, the calculations performed in the proof of Proposition 3.1 show that  $Q(\partial_i^u \wedge \partial_j^u) = 0$ .  $\square$

#### 4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

PROPOSITION 4.1. *Let  $M = B \times_f F$  be a warped product, where  $B$  is a small open ball around  $(0, 0)$  in  $\mathbf{R}^2$ , where  $f(x, y) = x + y + xy + 1$  is positive, and where  $F = \mathbf{R}$ . Then  $M$  is not flat. Furthermore  $Q(\partial_x \wedge \partial_y) = 0$ ,  $Q(\partial_x \wedge \partial_z) = 0$ , and  $Q(\partial_y \wedge \partial_z) = 0$ .*

*Proof.* We use [O, p. 210, Proposition 42], to compute:

$$\begin{aligned} \langle R(\partial_x, \partial_y) \partial_x, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_x, \partial_z \rangle &= 0, \\ \langle R(\partial_y, \partial_z) \partial_y, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_z, \partial_y \rangle &= f. \end{aligned} \quad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.