Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON AN ASSERTION IN RIEMANN'S HABILITATIONSVORTRAG
Autor: Di SCALA, Antonio J.
Kapitel: 3. Curvature zero 2-planes in $S^a \times H^a \times T^b$
DOI: https://doi.org/10.5169/seals-65428

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 11.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

More generally let $n \geq 3$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, a simple counting argument then shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3. Curvature zero 2-planes in $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a, S^a, and T^b be spaces of constant sectional curvature -1, $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

Proposition 3.1. *Let $M(a,b) := S^a \times H^a \times T^b$ with the product metric, where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a,b)$ such that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$\begin{aligned} e_{2i-1} &:= \tfrac{u_i+v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \tfrac{u_i-v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j && \text{for } 1 \leq j \leq b. \end{aligned}$$

The $\{e_k\}$ for $1 \leq k \leq 2a + b$ form a local orthonormal frame for the tangent space of $M(a,b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j)\, w_j, u_i\rangle = 0$, $\langle R(v_i, w_j)\, w_j, v_i\rangle = 0$, and $\langle R(v_i, w_j)\, w_j, v_i\rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2})\, u_{i_2}, u_{i_1}\rangle = +1$ and $\langle R(v_{i_1}, v_{i_2})\, v_{i_1}, v_{i_2}\rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \leq 2a$ and $j \leq 2a$ by computing:

$$\begin{aligned} &\langle R(e_1, e_2)\, e_2, e_1\rangle = 0, \\ &\langle R(e_1, e_3)\, e_3, e_1\rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)\, u_2, u_1\rangle + \langle R(v_1, v_2)\, v_2, v_1\rangle\} = 0, \\ &\langle R(e_1, e_4)\, e_4, e_1\rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)\, u_2, u_1\rangle + (-1)^2\langle R(v_1, v_2)\, v_2, v_1\rangle\} = 0, \text{ etc.} \quad \square \end{aligned}$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If $(x_1, \ldots, x_n)$ is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_j}$.

PROPOSITION 3.2. *Let* $M(2,b) := S^2 \times H^2 \times T^b$. *There exist local coordinates* $(u_1, \ldots, u_{4+b})$ *on* $M(2,b)$ *such that* $Q(\partial_i^u \wedge \partial_j^u) = 0$ *for* $1 \leq i < j \leq 4+b$.

Let ω be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

LEMMA 3.3. *Let* M^n *be an orientable Riemannian manifold. Then around each point there exists a coordinate system* $\{x_1, \ldots, x_n\}$ *such that* $\omega(\partial_1^x, \ldots, \partial_n^x) = 1$.

Proof of Proposition 3.2. We use Lemma 3.3 to find local coordinates (x_1, x_2) and (y_1, y_2) on S^2 and H^2 such that $\omega(\partial_1^x, \partial_2^x) = 1$ and $\omega(\partial_1^y, \partial_2^y) = 1$. Let $(z_1, \ldots, z_b)$ be the usual flat coordinates on T^b. Define local coordinates on $S^2 \times H^2 \times T^b$ by:

$$u_1 := x_1 + y_1\,, \qquad u_2 := x_1 - y_1\,, \qquad u_3 := x_2 + y_2\,, \qquad u_4 := x_2 - y_2\,,$$

and $u_{k+4} = w_k$ for $1 \leq k \leq b$. We then have

$$\partial_1^u = \partial_1^x + \partial_1^y\,, \qquad \partial_2^u = \partial_1^x - \partial_1^y\,, \qquad \partial_3^u = \partial_2^x + \partial_2^y\,, \qquad \partial_4^u = \partial_2^x + \partial_2^y,$$

and $\partial_{4+k}^u = \partial_k^w$ for $k > 0$. If N is a Riemann surface with constant sectional curvature ϵ, then $\langle R(x,y)\,y, x\rangle = \epsilon\,\omega(x,y)$. Thus, the calculations performed in the proof of Proposition 3.1 show that $Q(\partial_i^u \wedge \partial_i^v) = 0$. $\square$

4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

PROPOSITION 4.1. *Let* $M = B \times_f F$ *be a warped product, where* B *is a small open ball around* $(0,0)$ *in* $\mathbf{R}^2$, *where* $f(x,y) = x + y + xy + 1$ *is positive, and where* $F = \mathbf{R}$. *Then* M *is not flat. Furthermore* $Q(\partial_x \wedge \partial_y) = 0$, $Q(\partial_x \wedge \partial_z) = 0$, *and* $Q(\partial_y \wedge \partial_z) = 0$.

Proof. We use [O, p. 210, Proposition 42], to compute:

$$\langle R(\partial_x, \partial_y)\,\partial_x, \partial_z\rangle = 0, \qquad \langle R(\partial_x, \partial_z)\,\partial_x, \partial_z\rangle = 0,$$
$$\langle R(\partial_y, \partial_z)\,\partial_y, \partial_z\rangle = 0, \qquad \langle R(\partial_x, \partial_z)\,\partial_z, \partial_y\rangle = f. \qquad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.