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More generally let $n \geq 3$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V . If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, a simple counting argument then shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3. CURVATURE ZERO 2-PLANES IN $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a , S^a , and T^b be spaces of constant sectional curvature -1 , $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

PROPOSITION 3.1. *Let $M(a, b) := S^a \times H^a \times T^b$ with the product metric, where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a, b)$ such that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$\begin{aligned} e_{2i-1} &:= \frac{u_i + v_i}{\sqrt{2}} & \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \frac{u_i - v_i}{\sqrt{2}} & \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j & \text{for } 1 \leq j \leq b. \end{aligned}$$

The $\{e_k\}$ for $1 \leq k \leq 2a + b$ form a local orthonormal frame for the tangent space of $M(a, b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j) w_j, u_i \rangle = 0$, $\langle R(v_i, w_j) w_j, v_i \rangle = 0$, and $\langle R(v_i, w_j) w_j, v_i \rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2}) u_{i_2}, u_{i_1} \rangle = +1$ and $\langle R(v_{i_1}, v_{i_2}) v_{i_2}, v_{i_1} \rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \leq 2a$ and $j \leq 2a$ by computing:

$$\begin{aligned} \langle R(e_1, e_2) e_2, e_1 \rangle &= 0, \\ \langle R(e_1, e_3) e_3, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \\ \langle R(e_1, e_4) e_4, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \text{ etc. } \square \end{aligned}$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If (x_1, \dots, x_n) is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_i}$.

PROPOSITION 3.2. *Let $M(2, b) := S^2 \times H^2 \times T^b$. There exist local coordinates (u_1, \dots, u_{4+b}) on $M(2, b)$ such that $Q(\partial_i^u \wedge \partial_j^u) = 0$ for $1 \leq i < j \leq 4 + b$.*

Let ω be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

LEMMA 3.3. *Let M^n be an orientable Riemannian manifold. Then around each point there exists a coordinate system $\{x_1, \dots, x_n\}$ such that $\omega(\partial_1^x, \dots, \partial_n^x) = 1$.*

Proof of Proposition 3.2. We use Lemma 3.3 to find local coordinates (x_1, x_2) and (y_1, y_2) on S^2 and H^2 such that $\omega(\partial_1^x, \partial_2^x) = 1$ and $\omega(\partial_1^y, \partial_2^y) = 1$. Let (z_1, \dots, z_b) be the usual flat coordinates on T^b . Define local coordinates on $S^2 \times H^2 \times T^b$ by:

$$u_1 := x_1 + y_1, \quad u_2 := x_1 - y_1, \quad u_3 := x_2 + y_2, \quad u_4 := x_2 - y_2,$$

and $u_{k+4} = z_k$ for $1 \leq k \leq b$. We then have

$$\partial_1^u = \partial_1^x + \partial_1^y, \quad \partial_2^u = \partial_1^x - \partial_1^y, \quad \partial_3^u = \partial_2^x + \partial_2^y, \quad \partial_4^u = \partial_2^x - \partial_2^y,$$

and $\partial_{4+k}^u = \partial_k^z$ for $k > 0$. If N is a Riemann surface with constant sectional curvature ϵ , then $\langle R(x, y)y, x \rangle = \epsilon \omega(x, y)$. Thus, the calculations performed in the proof of Proposition 3.1 show that $Q(\partial_i^u \wedge \partial_j^u) = 0$. \square

4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

PROPOSITION 4.1. *Let $M = B \times_f F$ be a warped product, where B is a small open ball around $(0, 0)$ in \mathbf{R}^2 , where $f(x, y) = x + y + xy + 1$ is positive, and where $F = \mathbf{R}$. Then M is not flat. Furthermore $Q(\partial_x \wedge \partial_y) = 0$, $Q(\partial_x \wedge \partial_z) = 0$, and $Q(\partial_y \wedge \partial_z) = 0$.*

Proof. We use [O, p. 210, Proposition 42], to compute:

$$\begin{aligned} \langle R(\partial_x, \partial_y) \partial_x, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_x, \partial_z \rangle &= 0, \\ \langle R(\partial_y, \partial_z) \partial_y, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_z, \partial_y \rangle &= f. \end{aligned} \quad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.