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More generally let  $n \geq 3$  and let  $\{e_1, \dots, e_n\}$  be an orthonormal basis for  $V$ . If we impose the condition that  $Q_K(e_i \wedge e_j) = 0$  with  $i < j$ , then we have imposed  $\frac{n(n-1)}{2}$  conditions. Since the dimension of the space of algebraic curvature tensors is  $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$ , a simple counting argument then shows there are non-trivial algebraic curvatures with  $Q_K(e_i \wedge e_j) = 0$  for  $i < j$ ; thus Assertion 1.1 fails in the algebraic setting.

### 3. CURVATURE ZERO 2-PLANES IN $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let  $H^a$ ,  $S^a$ , and  $T^b$  be spaces of constant sectional curvature  $-1$ ,  $+1$ , and  $0$  where  $a \geq 2$ . We begin by studying orthonormal frame fields.

**PROPOSITION 3.1.** *Let  $M(a, b) := S^a \times H^a \times T^b$  with the product metric, where  $a \geq 2$ . There exists a local orthonormal frame  $\{e_i\}$  for the tangent bundle of  $M(a, b)$  such that  $Q(e_i \wedge e_j) = 0$  for  $1 \leq i < j \leq 2a + b$ .*

*Proof.* Let  $\{u_i\}$  and  $\{v_i\}$  be local orthonormal frames for the tangent bundles of  $S^a$  and  $H^a$  for  $1 \leq i \leq a$ . Let  $\{w_j\}$  be a local orthonormal frame for the tangent bundle of  $T^b$  for  $1 \leq j \leq b$ . Define

$$\begin{aligned} e_{2i-1} &:= \frac{u_i + v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \frac{u_i - v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j && \text{for } 1 \leq j \leq b. \end{aligned}$$

The  $\{e_k\}$  for  $1 \leq k \leq 2a + b$  form a local orthonormal frame for the tangent space of  $M(a, b) := S^a \times H^a \times T^b$ . We have  $\langle R(u_i, w_j) w_j, u_i \rangle = 0$ ,  $\langle R(v_i, w_j) w_j, v_i \rangle = 0$ , and  $\langle R(v_i, w_j) w_j, v_i \rangle = 0$ . Thus  $Q(e_i \wedge e_j) = 0$  if either  $i > 2a$  or  $j > 2a$ . We also have  $\langle R(u_{i_1}, u_{i_2}) u_{i_2}, u_{i_1} \rangle = +1$  and  $\langle R(v_{i_1}, v_{i_2}) v_{i_1}, v_{i_2} \rangle = -1$  for  $i_1 < i_2$ . We can show that  $Q(e_i \wedge e_j) = 0$  for  $i \leq 2a$  and  $j \leq 2a$  by computing:

$$\begin{aligned} \langle R(e_1, e_2) e_2, e_1 \rangle &= 0, \\ \langle R(e_1, e_3) e_3, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \\ \langle R(e_1, e_4) e_4, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \text{ etc. } \square \end{aligned}$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If  $(x_1, \dots, x_n)$  is a system of local coordinates, set  $\partial_i^x := \frac{\partial}{\partial x_i}$ .

**PROPOSITION 3.2.** *Let  $M(2, b) := S^2 \times H^2 \times T^b$ . There exist local coordinates  $(u_1, \dots, u_{4+b})$  on  $M(2, b)$  such that  $Q(\partial_i^u \wedge \partial_j^u) = 0$  for  $1 \leq i < j \leq 4 + b$ .*

Let  $\omega$  be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

**LEMMA 3.3.** *Let  $M^n$  be an orientable Riemannian manifold. Then around each point there exists a coordinate system  $\{x_1, \dots, x_n\}$  such that  $\omega(\partial_1^x, \dots, \partial_n^x) = 1$ .*

*Proof of Proposition 3.2.* We use Lemma 3.3 to find local coordinates  $(x_1, x_2)$  and  $(y_1, y_2)$  on  $S^2$  and  $H^2$  such that  $\omega(\partial_1^x, \partial_2^x) = 1$  and  $\omega(\partial_1^y, \partial_2^y) = 1$ . Let  $(z_1, \dots, z_b)$  be the usual flat coordinates on  $T^b$ . Define local coordinates on  $S^2 \times H^2 \times T^b$  by:

$$u_1 := x_1 + y_1, \quad u_2 := x_1 - y_1, \quad u_3 := x_2 + y_2, \quad u_4 := x_2 - y_2,$$

and  $u_{k+4} = w_k$  for  $1 \leq k \leq b$ . We then have

$$\partial_1^u = \partial_1^x + \partial_1^y, \quad \partial_2^u = \partial_1^x - \partial_1^y, \quad \partial_3^u = \partial_2^x + \partial_2^y, \quad \partial_4^u = \partial_2^x - \partial_2^y,$$

and  $\partial_{4+k}^u = \partial_k^w$  for  $k > 0$ . If  $N$  is a Riemann surface with constant sectional curvature  $\epsilon$ , then  $\langle R(x, y) y, x \rangle = \epsilon \omega(x, y)$ . Thus, the calculations performed in the proof of Proposition 3.1 show that  $Q(\partial_i^u \wedge \partial_j^u) = 0$ .  $\square$

#### 4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

**PROPOSITION 4.1.** *Let  $M = B \times_f F$  be a warped product, where  $B$  is a small open ball around  $(0, 0)$  in  $\mathbf{R}^2$ , where  $f(x, y) = x + y + xy + 1$  is positive, and where  $F = \mathbf{R}$ . Then  $M$  is not flat. Furthermore  $Q(\partial_x \wedge \partial_y) = 0$ ,  $Q(\partial_x \wedge \partial_z) = 0$ , and  $Q(\partial_y \wedge \partial_z) = 0$ .*

*Proof.* We use [O, p. 210, Proposition 42], to compute:

$$\begin{aligned} \langle R(\partial_x, \partial_y) \partial_x, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_x, \partial_z \rangle &= 0, \\ \langle R(\partial_y, \partial_z) \partial_y, \partial_z \rangle &= 0, & \langle R(\partial_x, \partial_z) \partial_z, \partial_y \rangle &= f. \end{aligned} \quad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.