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More generally let $n \geq 3$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, a simple counting argument then shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3. Curvature zero 2-planes in $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a, S^a, and T^b be spaces of constant sectional curvature -1, $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

PROPOSITION 3.1. *Let $M(a,b) := S^a \times H^a \times T^b$ with the product metric, where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a,b)$ such that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$\begin{aligned} e_{2i-1} &:= \tfrac{u_i + v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \tfrac{u_i - v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j && \text{for } 1 \leq j \leq b. \end{aligned}$$

The $\{e_k\}$ for $1 \leq k \leq 2a + b$ form a local orthonormal frame for the tangent space of $M(a,b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j)\, w_j, u_i \rangle = 0$, $\langle R(v_i, w_j)\, w_j, v_i \rangle = 0$, and $\langle R(v_i, w_j)\, w_j, v_i \rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2})\, u_{i_2}, u_{i_1} \rangle = +1$ and $\langle R(v_{i_1}, v_{i_2})\, v_{i_1}, v_{i_2} \rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \leq 2a$ and $j \leq 2a$ by computing:

$$\begin{aligned} &\langle R(e_1, e_2)\, e_2, e_1 \rangle = 0, \\ &\langle R(e_1, e_3)\, e_3, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)\, u_2, u_1 \rangle + \langle R(v_1, v_2)\, v_2, v_1 \rangle\} = 0, \\ &\langle R(e_1, e_4)\, e_4, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)\, u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2)\, v_2, v_1 \rangle\} = 0, \text{ etc.} \end{aligned}$$ □

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If $(x_1, \ldots, x_n)$ is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_j}$.

PROPOSITION 3.2. *Let* $M(2,b) := S^2 \times H^2 \times T^b$. *There exist local coordinates* $(u_1, \dots, u_{4+b})$ *on* $M(2,b)$ *such that* $Q(\partial_i^u \wedge \partial_j^u) = 0$ *for* $1 \le i < j \le 4+b$.

Let ω be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to [K, p. 6] for details:

LEMMA 3.3. *Let* M^n *be an orientable Riemannian manifold. Then around each point there exists a coordinate system* $\{x_1, \dots, x_n\}$ *such that* $\omega(\partial_1^x, \dots, \partial_n^x) = 1$.

Proof of Proposition 3.2. We use Lemma 3.3 to find local coordinates (x_1, x_2) and (y_1, y_2) on S^2 and H^2 such that $\omega(\partial_1^x, \partial_2^x) = 1$ and $\omega(\partial_1^y, \partial_2^y) = 1$. Let $(z_1, \dots, z_b)$ be the usual flat coordinates on T^b. Define local coordinates on $S^2 \times H^2 \times T^b$ by:

$$u_1 := x_1 + y_1, \qquad u_2 := x_1 - y_1, \qquad u_3 := x_2 + y_2, \qquad u_4 := x_2 - y_2,$$

and $u_{k+4} = w_k$ for $1 \le k \le b$. We then have

$$\partial_1^u = \partial_1^x + \partial_1^y, \qquad \partial_2^u = \partial_1^x - \partial_1^y, \qquad \partial_3^u = \partial_2^x + \partial_2^y, \qquad \partial_4^u = \partial_2^x + \partial_2^y,$$

and $\partial_{4+k}^u = \partial_k^w$ for $k > 0$. If N is a Riemann surface with constant sectional curvature ϵ, then $\langle R(x,y)y, x\rangle = \epsilon\,\omega(x,y)$. Thus, the calculations performed in the proof of Proposition 3.1 show that $Q(\partial_i^u \wedge \partial_i^v) = 0$. □

4. CURVATURE ZERO 2-PLANES IN WARPED PRODUCTS

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, p. 210].

PROPOSITION 4.1. *Let* $M = B \times_f F$ *be a warped product, where* B *is a small open ball around* $(0,0)$ *in* $\mathbf{R}^2$, *where* $f(x,y) = x + y + xy + 1$ *is positive, and where* $F = \mathbf{R}$. *Then* M *is not flat. Furthermore* $Q(\partial_x \wedge \partial_y) = 0$, $Q(\partial_x \wedge \partial_z) = 0$, *and* $Q(\partial_y \wedge \partial_z) = 0$.

Proof. We use [O, p. 210, Proposition 42], to compute:

$$\langle R(\partial_x, \partial_y)\,\partial_x, \partial_z\rangle = 0, \qquad \langle R(\partial_x, \partial_z)\,\partial_x, \partial_z\rangle = 0,$$
$$\langle R(\partial_y, \partial_z)\,\partial_y, \partial_z\rangle = 0, \qquad \langle R(\partial_x, \partial_z)\,\partial_z, \partial_y\rangle = f. \qquad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.