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## 2. AN ALGEBRAIC EXAMPLE

Let  $V$  be an  $n$ -dimensional real vector space and let  $\langle \cdot, \cdot \rangle$  be a positive definite inner product defined on  $V$ . A bilinear  $R: V \times V \rightarrow \text{End}(V)$  is called an *algebraic curvature tensor* if it has the following three properties:

- (1)  $\langle R(x, y)z, w \rangle = -\langle R(y, x)z, w \rangle$
- (2)  $\langle R(x, y)z, w \rangle = -\langle R(x, y)w, z \rangle$
- (3)  $\langle R(x, y)z, w \rangle + \langle R(y, z)x, w \rangle + \langle R(z, x)y, w \rangle = 0$

These three properties then imply the following symmetry property

$$\langle R(x, y)z, w \rangle = \langle R(z, w)x, y \rangle;$$

see [KN, p. 198] or [Sp1, p. 4D-17]) for details. We can also identify the space of algebraic curvature tensors with the space  $K$  of symmetric endomorphisms of the second exterior product  $\bigwedge^2(V)$  such that:

$$(4) \quad \langle K(x \wedge y), z \wedge w \rangle + \langle K(y \wedge z), x \wedge w \rangle + \langle K(z \wedge x), y \wedge w \rangle = 0.$$

Here the inner product on  $\bigwedge^2(V)$  is induced from the inner product on  $V$ . We say that a collection of 2-dimensional subspaces are linearly independent if the associated elements of  $\bigwedge^2(V)$  are linearly independent in  $\bigwedge^2(V)$ . For example, let  $\{e_1, \dots, e_n\}$  be a basis of  $V$ . Then the 2-subspaces spanned by  $\{e_i, e_j\}_{i \neq j}$  are independent. The bi-quadratic tensor  $\langle R(x, y)y, x \rangle$  determines  $R$ ; we refer to [KN, p. 198] for the proof of the following result:

**PROPOSITION 2.1.** *Let  $R$  be an algebraic curvature tensor such that*

$$\langle R(x, y)y, x \rangle = 0 \quad \text{for all } x \text{ and } y.$$

*Then  $R = 0$ .*

The space of curvature tensors has dimension  $\frac{n^2(n^2-1)}{12}$ ; see for example M. Berger [B, p. 63]. Thus, if  $n = 3$  then equations (3) and (4) follow from equations (1) and (2). Let  $\{e_1, e_2, e_3\}$  be an orthonormal basis for  $V$ . We define a symmetric endomorphism  $K$  of  $\bigwedge^2(V)$  by:

$$K(e_1 \wedge e_2) = e_3 \wedge e_1, \quad K(e_2 \wedge e_3) = 0, \quad K(e_3 \wedge e_1) = e_1 \wedge e_2.$$

Note that  $K$  is a non-trivial algebraic curvature tensor with the following three vanishing sectional curvatures:

$$Q_K(e_1 \wedge e_2) = Q_K(e_2 \wedge e_3) = Q_K(e_3 \wedge e_1) = 0.$$

More generally let  $n \geq 3$  and let  $\{e_1, \dots, e_n\}$  be an orthonormal basis for  $V$ . If we impose the condition that  $Q_K(e_i \wedge e_j) = 0$  with  $i < j$ , then we have imposed  $\frac{n(n-1)}{2}$  conditions. Since the dimension of the space of algebraic curvature tensors is  $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$ , a simple counting argument then shows there are non-trivial algebraic curvatures with  $Q_K(e_i \wedge e_j) = 0$  for  $i < j$ ; thus Assertion 1.1 fails in the algebraic setting.

### 3. CURVATURE ZERO 2-PLANES IN $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let  $H^a$ ,  $S^a$ , and  $T^b$  be spaces of constant sectional curvature  $-1$ ,  $+1$ , and  $0$  where  $a \geq 2$ . We begin by studying orthonormal frame fields.

**PROPOSITION 3.1.** *Let  $M(a, b) := S^a \times H^a \times T^b$  with the product metric, where  $a \geq 2$ . There exists a local orthonormal frame  $\{e_i\}$  for the tangent bundle of  $M(a, b)$  such that  $Q(e_i \wedge e_j) = 0$  for  $1 \leq i < j \leq 2a + b$ .*

*Proof.* Let  $\{u_i\}$  and  $\{v_i\}$  be local orthonormal frames for the tangent bundles of  $S^a$  and  $H^a$  for  $1 \leq i \leq a$ . Let  $\{w_j\}$  be a local orthonormal frame for the tangent bundle of  $T^b$  for  $1 \leq j \leq b$ . Define

$$\begin{aligned} e_{2i-1} &:= \frac{u_i + v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \frac{u_i - v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j && \text{for } 1 \leq j \leq b. \end{aligned}$$

The  $\{e_k\}$  for  $1 \leq k \leq 2a + b$  form a local orthonormal frame for the tangent space of  $M(a, b) := S^a \times H^a \times T^b$ . We have  $\langle R(u_i, w_j) w_j, u_i \rangle = 0$ ,  $\langle R(v_i, w_j) w_j, v_i \rangle = 0$ , and  $\langle R(v_i, w_j) w_j, v_i \rangle = 0$ . Thus  $Q(e_i \wedge e_j) = 0$  if either  $i > 2a$  or  $j > 2a$ . We also have  $\langle R(u_{i_1}, u_{i_2}) u_{i_2}, u_{i_1} \rangle = +1$  and  $\langle R(v_{i_1}, v_{i_2}) v_{i_1}, v_{i_2} \rangle = -1$  for  $i_1 < i_2$ . We can show that  $Q(e_i \wedge e_j) = 0$  for  $i \leq 2a$  and  $j \leq 2a$  by computing:

$$\begin{aligned} \langle R(e_1, e_2) e_2, e_1 \rangle &= 0, \\ \langle R(e_1, e_3) e_3, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \\ \langle R(e_1, e_4) e_4, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \text{ etc. } \square \end{aligned}$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If  $(x_1, \dots, x_n)$  is a system of local coordinates, set  $\partial_i^x := \frac{\partial}{\partial x_i}$ .