Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON AN ASSERTION IN RIEMANN'S HABILITATIONSVORTRAG
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Kapitel: 2. An algebraic example
DOI: https://doi.org/10.5169/seals-65428

2. AN ALGEBRAIC EXAMPLE

Let V be an n-dimensional real vector space and let $\langle\ ,\ \rangle$ be a positive definite inner product defined on V. A bilinear $R\colon V \times V \to \mathrm{End}(V)$ is called an *algebraic curvature tensor* if it has the following three properties:

(1) $$\langle R(x,y)\,z, w\rangle = -\langle R(y,x)\,z, w\rangle$$

(2) $$\langle R(x,y)\,z, w\rangle = -\langle R(x,y)\,w, z\rangle$$

(3) $$\langle R(x,y)\,z, w\rangle + \langle R(y,z)\,x, w\rangle + \langle R(z,x)\,y, w\rangle = 0$$

These three properties then imply the following symmetry property

$$\langle R(x,y)\,z, w\rangle = \langle R(z,w)\,x, y\rangle\,;$$

see [KN, p. 198] or [Sp1, p. 4D-17]) for details. We can also identify the space of algebraic curvature tensors with the space K of symmetric endomorphisms of the second exterior product $\bigwedge^2(V)$ such that:

(4) $$\langle K(x \wedge y), z \wedge w\rangle + \langle K(y \wedge z), x \wedge w\rangle + \langle K(z \wedge x), y \wedge w\rangle = 0\,.$$

Here the inner product on $\bigwedge^2(V)$ is induced from the inner product on V. We say that a collection of 2-dimensional subspaces are linearly independent if the associated elements of $\bigwedge^2(V)$ are linearly independent in $\bigwedge^2(V)$. For example, let $\{e_1, \dots, e_n\}$ be a basis of V. Then the 2-subspaces spanned by $\{e_i, e_j\}_{i \neq j}$ are independent. The bi-quadratic tensor $\langle R(x,y)\,y, x\rangle$ determines R; we refer to [KN, p. 198] for the proof of the following result:

PROPOSITION 2.1. *Let R be an algebraic curvature tensor such that*

$$\langle R(x,y)\,y, x\rangle = 0 \qquad \textit{for all } x \textit{ and } y.$$

Then $R = 0$.

The space of curvature tensors has dimension $\frac{n^2(n^2-1)}{12}$; see for example M. Berger [B, p. 63]. Thus, if $n = 3$ then equations (3) and (4) follow from equations (1) and (2). Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for V. We define a symmetric endomorphism K of $\bigwedge^2(V)$ by:

$$K(e_1 \wedge e_2) = e_3 \wedge e_1\,, \qquad K(e_2 \wedge e_3) = 0\,, \qquad K(e_3 \wedge e_1) = e_1 \wedge e_2\,.$$

Note that K is a non-trivial algebraic curvature tensor with the following three vanishing sectional curvatures:

$$Q_K(e_1 \wedge e_2) = Q_K(e_2 \wedge e_3) = Q_K(e_3 \wedge e_1) = 0\,.$$

More generally let $n \geq 3$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, a simple counting argument then shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3. Curvature zero 2-planes in $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a, S^a, and T^b be spaces of constant sectional curvature -1, $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

PROPOSITION 3.1. *Let $M(a,b) := S^a \times H^a \times T^b$ with the product metric, where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a,b)$ such that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$\begin{aligned} e_{2i-1} &:= \tfrac{u_i + v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \tfrac{u_i - v_i}{\sqrt{2}} && \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j && \text{for } 1 \leq j \leq b. \end{aligned}$$

The $\{e_k\}$ for $1 \leq k \leq 2a + b$ form a local orthonormal frame for the tangent space of $M(a,b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j) w_j, u_i \rangle = 0$, $\langle R(v_i, w_j) w_j, v_i \rangle = 0$, and $\langle R(v_i, w_j) w_j, v_i \rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2}) u_{i_2}, u_{i_1} \rangle = +1$ and $\langle R(v_{i_1}, v_{i_2}) v_{i_1}, v_{i_2} \rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \leq 2a$ and $j \leq 2a$ by computing:

$$\langle R(e_1, e_2) e_2, e_1 \rangle = 0,$$
$$\langle R(e_1, e_3) e_3, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2) u_2, u_1 \rangle + \langle R(v_1, v_2) v_2, v_1 \rangle\} = 0,$$
$$\langle R(e_1, e_4) e_4, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2) u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2) v_2, v_1 \rangle\} = 0, \text{ etc.} \quad \square$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If $(x_1, \ldots, x_n)$ is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_j}$.