

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON AN ASSERTION IN RIEMANN'S HABILITATIONSVORTRAG
Autor: Di SCALA, Antonio J.
Kapitel: 2. An algebraic example
DOI: <https://doi.org/10.5169/seals-65428>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2. AN ALGEBRAIC EXAMPLE

Let V be an n -dimensional real vector space and let $\langle \cdot, \cdot \rangle$ be a positive definite inner product defined on V . A bilinear $R: V \times V \rightarrow \text{End}(V)$ is called an *algebraic curvature tensor* if it has the following three properties:

- (1) $\langle R(x, y)z, w \rangle = -\langle R(y, x)z, w \rangle$
- (2) $\langle R(x, y)z, w \rangle = -\langle R(x, y)w, z \rangle$
- (3) $\langle R(x, y)z, w \rangle + \langle R(y, z)x, w \rangle + \langle R(z, x)y, w \rangle = 0$

These three properties then imply the following symmetry property

$$\langle R(x, y)z, w \rangle = \langle R(z, w)x, y \rangle;$$

see [KN, p. 198] or [Sp1, p. 4D-17]) for details. We can also identify the space of algebraic curvature tensors with the space K of symmetric endomorphisms of the second exterior product $\wedge^2(V)$ such that:

- (4) $\langle K(x \wedge y), z \wedge w \rangle + \langle K(y \wedge z), x \wedge w \rangle + \langle K(z \wedge x), y \wedge w \rangle = 0$.

Here the inner product on $\wedge^2(V)$ is induced from the inner product on V . We say that a collection of 2-dimensional subspaces are linearly independent if the associated elements of $\wedge^2(V)$ are linearly independent in $\wedge^2(V)$. For example, let $\{e_1, \dots, e_n\}$ be a basis of V . Then the 2-subspaces spanned by $\{e_i, e_j\}_{i \neq j}$ are independent. The bi-quadratic tensor $\langle R(x, y)y, x \rangle$ determines R ; we refer to [KN, p. 198] for the proof of the following result:

PROPOSITION 2.1. *Let R be an algebraic curvature tensor such that*

$$\langle R(x, y)y, x \rangle = 0 \quad \text{for all } x \text{ and } y.$$

Then $R = 0$.

The space of curvature tensors has dimension $\frac{n^2(n^2-1)}{12}$; see for example M. Berger [B, p. 63]. Thus, if $n = 3$ then equations (3) and (4) follow from equations (1) and (2). Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for V . We define a symmetric endomorphism K of $\wedge^2(V)$ by:

$$K(e_1 \wedge e_2) = e_3 \wedge e_1, \quad K(e_2 \wedge e_3) = 0, \quad K(e_3 \wedge e_1) = e_1 \wedge e_2.$$

Note that K is a non-trivial algebraic curvature tensor with the following three vanishing sectional curvatures:

$$Q_K(e_1 \wedge e_2) = Q_K(e_2 \wedge e_3) = Q_K(e_3 \wedge e_1) = 0.$$

More generally let $n \geq 3$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V . If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, a simple counting argument then shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3. CURVATURE ZERO 2-PLANES IN $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a , S^a , and T^b be spaces of constant sectional curvature -1 , $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

PROPOSITION 3.1. *Let $M(a, b) := S^a \times H^a \times T^b$ with the product metric, where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a, b)$ such that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$\begin{aligned} e_{2i-1} &:= \frac{u_i + v_i}{\sqrt{2}} & \text{for } 1 \leq i \leq a, \\ e_{2i} &:= \frac{u_i - v_i}{\sqrt{2}} & \text{for } 1 \leq i \leq a, \\ e_{2a+j} &:= w_j & \text{for } 1 \leq j \leq b. \end{aligned}$$

The $\{e_k\}$ for $1 \leq k \leq 2a + b$ form a local orthonormal frame for the tangent space of $M(a, b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j) w_j, u_i \rangle = 0$, $\langle R(v_i, w_j) w_j, v_i \rangle = 0$, and $\langle R(v_i, w_j) w_j, v_i \rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2}) u_{i_2}, u_{i_1} \rangle = +1$ and $\langle R(v_{i_1}, v_{i_2}) v_{i_1}, v_{i_2} \rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \leq 2a$ and $j \leq 2a$ by computing:

$$\begin{aligned} \langle R(e_1, e_2) e_2, e_1 \rangle &= 0, \\ \langle R(e_1, e_3) e_3, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \\ \langle R(e_1, e_4) e_4, e_1 \rangle &= \frac{1}{4} \{ \langle R(u_1, u_2) u_2, u_1 \rangle + (-1)^2 \langle R(v_1, v_2) v_2, v_1 \rangle \} = 0, \text{ etc. } \square \end{aligned}$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If (x_1, \dots, x_n) is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_i}$.