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ON AN ASSERTION IN RIEMANN'S HABILITATIONSVORTRAG

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ABSTRACT. We study an assertion in Riemann's Habilitation Lecture of 1854. Namely, the determination of the metric given $n\frac{n-1}{2}$ sectional curvatures.

1. INTRODUCTION

Modern differential geometry was born with Riemann's Habilitation Lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" (*On the Hypotheses which lie at the Foundations of Geometry*) of 1854 at Göttingen [R], [We]. In this lecture Riemann defines the curvature tensor R. One says that M is *flat* if M is locally isometric to $\mathbf{R}^n$ with the usual metric; the tensor R vanishes if and only if the metric is flat. M. Spivak [Sp1] translates Riemann's Lecture and explains it in modern terms. Let

$$Q(X,Y) := \frac{\langle R(X,Y)\,Y, X\rangle}{|X \wedge Y|^2}$$

be the sectional curvature. Spivak [Sp1, p. 4B-25], [Sp2, p. 176] makes the following

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ASSERTION 1.1. *If M is n-dimensional and if $Q = 0$ for $n\frac{n-1}{2}$ independent 2-dimensional subspaces of each M_q, then M is flat.*

It is well known that the metric is flat if and only if the sectional curvature Q vanishes identically. The number $n\frac{n-1}{2}$ of Assertion 1.1 is "deduced" from the following "counting argument" given by Riemann: the metric $ds^2 = \sum g_{ij}\, dx_i\, dx_j$ contains $\frac{n(n+1)}{2}$ functions while a new coordinate system involves only n functions, so that we can change only n of the g_{ij}, leaving $\frac{n(n-1)}{2}$ other functions which depend on the metric; thus there should be some set of $\frac{n(n-1)}{2}$ functions which will determine the metric completely (see [Di, p. 198], [Sp1, p. 4B-4]). We quote from the original text as follows [We], [R]:

> "... wenn also das Krümmungsmaß in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen gegeben wird, so werden daraus die Maßverhältnisse der Mannigfaltigkeit sich bestimmen lassen, wofern nur zwischen diesen Werthen keine identischen Relationen stattfinden, was in der That, allgemein zu reden, nicht der Fall ist."
>
> "... es reicht aber nach der frühern Untersuchung, um die Maßverhältnisse zu bestimmen, hin zu wissen, daß es in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen, deren Krümmungsmaße von einander unabhängig sind, Null sei."

We remark that this text is omitted by Hermann Weyl in his discussion of Riemann's ideas. Relating the curvature tensor to the metric is a very classical subject and we refer to [Ku, Ya, B] for further details.

In this note we construct several families of counter-examples to Assertion 1.1. In § 2 we discuss the space of algebraic curvature tensors and construct an algebraic curvature tensor in dimension 3 which has vanishing sectional curvature on three independent 2-planes: this shows that Assertion 1.1 is not an algebraic consequence of the curvature tensor identities. Let H^2, S^2 and T^k denote the hyperbolic plane, the sphere and the torus with the metrics of constant curvature -1, 1, and 0. Give $M = S^2 \times H^2 \times T^k$ the product metric; this manifold is not flat. In § 3 we construct local orthonormal frames $\{e_i\}$ and local coordinate frames ∂_i for the tangent bundle such that the sectional curvatures $Q(e_i, e_j)$ and $Q(\partial_i, \partial_j)$ vanish for $i \neq j$. Again this shows Assertion 1.1 is false. Finally, in § 4, we use warped products to construct still other examples of non-flat metrics which are counter-examples to Assertion 1.1. It is a pleasant task to thank Professors V. Cortez and P. Gilkey for helpful discussions concerning these matters.