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## 5. INVERSION AND MODULARITY

Since  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \subseteq \text{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$ , the inverse  $\phi^{-1}$  of a nondegenerate  $\phi \in \text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is well defined and lies in  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ . By Cramer's rule inversion is a rational map from  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  to  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ , more precisely there is a homogeneous polynomial map  $P: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  such that  $\phi^P \phi = \det(\phi) \cdot \text{id}_{\mathcal{V}}$ . Viewing this as an identity of matrices with polynomial entries, one might cancel out the greatest common divisor of all occurring entries and get new polynomial maps  $p: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  and  $d: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \mathbf{Q}$  with  $\phi^p \phi = d(\phi) \cdot \text{id}_{\mathcal{V}}$ . The properties of the map  $p$  have not been studied in this generality. The aim here is to investigate the simplest case, where  $p$  is homogeneous of degree 1, i.e. a  $\mathbf{Q}$ -linear map  $\iota$ , as it is called in the sequel. Of course, the same analysis can be done with  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ . The question whether such a  $\iota$  is an equivalence, will be treated later in this section.

**DEFINITION 5.1.** Let  $R$  be one of  $\mathbf{Z}$  or  $\mathbf{Q}$ . Then  $\text{Bil}_{\Lambda_R}(L_R)$  is called *special* if there is an  $R$ -linear map  $\iota: \text{Bil}_{\Lambda_R}(L_R) \rightarrow \text{Bil}_{\Lambda_R}(L_R^*)$  and a quadratic form  $q: \text{Bil}_{\Lambda_R}(L_R) \rightarrow R$  such that for any nondegenerate  $\phi \in \text{Bil}_{\Lambda_R}(L_R)$  one has  $\phi^\iota \phi = q(\phi) \text{id}_{L_R}$ . Analogous definitions hold for  $\text{Bil}_{\Lambda_R}^+(L_R)$ .

**EXAMPLE 5.2.**

(i) One-dimensional lattices of covariant forms are special for trivial reasons.

(ii) If  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$  is two-dimensional, then it is special. This is because  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$  can be viewed as a free  $Z(\mathcal{A})$ -module and for two-dimensional algebras  $\mathcal{B}$  one has a canonical automorphism  $\kappa$  of  $\mathcal{B}$  such that  $b^\kappa = n(b)b^{-1}$  for all  $b \in \mathcal{B}^*$ , where  $n: \mathcal{B} \rightarrow F$  is the norm map with respect to the regular representation. (Note that  $Z(\mathcal{A}) = \text{End}_{\mathcal{A}}(\mathcal{V})$  in the present situation.)

(iii) If  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is two-dimensional then it is special. This is because  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  can be viewed as a free  $Z(\mathcal{A})^+$ -module, where

$$Z(\mathcal{A})^+ := \{\varphi \in Z(\mathcal{A}) \mid \varphi^\circ = \varphi\}.$$

Here are some more interesting examples.

**PROPOSITION 5.3.** Let  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$  with  $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ . Then  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is special. In the first two cases also  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$  is special.

*Proof.* Define  $\mathcal{E} := \text{End}_{\mathcal{A}}(\mathcal{V}) \cong (e\mathcal{A}e)^{k \times k}$ , where  $e = e^\circ$  is a primitive  $^\circ$ -invariant idempotent of  $\mathcal{A}$  and  $k$  is defined by  $\mathcal{V} \cong (e\mathcal{A})^k$ . In particular, the positive involution  $^\circ$  on  $\mathcal{A}$  induces a positive involution  $^\bullet$  on  $\mathcal{E}$ ,  $(a_{ij})^\bullet := (a_{ij}^\circ)^{tr}$ , such that  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  can be identified with the subspace  $\mathcal{E}^+$  of the symmetric elements in the algebra  $(\mathcal{E}, \bullet)$  with involution. It suffices to prove that there exists a  $\mathbf{Q}$ -vector space automorphism of  $\mathcal{E}^+$ , also denoted by  $\iota$ , and a  $\mathbf{Q}$ -valued quadratic form on  $\mathcal{E}^+$ , also denoted by  $q$ , such that  $\phi^\iota \phi = q(\phi) 1_{\mathcal{E}}$ .

(i) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{R}^{2 \times 2}$ . Then  $\mathcal{E}$  is a quaternion algebra over  $\mathbf{Q}$ . Denote its canonical involution by  $\omega'$  and its reduced norm by  $n$ . Clearly,  $n$  is a quadratic form and  $\omega'(\phi) \phi = n(\phi) 1$  holds for all elements  $\phi \in \mathcal{E}$ . With  $\iota := \omega'|_{\mathcal{E}^+}$  and  $q := n|_{\mathcal{E}^+}$  one gets the desired formula.

(ii) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{C}^{2 \times 2}$ . Then  $\mathcal{E}$  is a quaternion algebra over the imaginary quadratic number field  $Z := Z(\mathcal{A})$ . Denote its canonical involution by  $\omega'$  and its reduced norm by  $n$ . The involution  $^\bullet$  induces the nontrivial Galois automorphism of  $(Z/\mathbf{Q})$ , and therefore one checks quite easily, using [Scha85] Theorem 11.2 (ii) of Chapter 8, that the norm  $n$  maps  $\mathcal{E}^+$  into  $\mathbf{Q}$ . Now one argues as in (i).

(iii) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{H}^{2 \times 2}$ . Then  $\mathcal{E} \cong D^{2 \times 2}$ , where  $D$  is a positive definite quaternion algebra over  $\mathbf{Q}$  (with canonical involution  $\omega'$ ). Indeed,  $\mathcal{E}$  carries an involution of the first kind and hence cannot be of index 4. Since  $^\bullet$  is a positive involution one sees from the proof of Theorem 13.3 of Chapter 8 in [Scha85] that  $x^\bullet = f^{-1} \bar{x}^{tr} f$  for all  $x \in \mathcal{E}$ , where  $f = \bar{f}^{tr} \in \mathcal{E}^*$  and  $\overline{(x_{ij})} = (\overline{x_{ij}})$  for all  $(x_{ij}) \in D^{2 \times 2} \cong \mathcal{E}$ . If  $(x_{ij}) \in \mathcal{E}$  is symmetric with respect to  $^{-tr}$  one checks

$$(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} \quad \text{with } \overline{x_{ii}} = x_{ii} \text{ for } i = 1, 2$$

$$\text{and } \begin{pmatrix} x_{22} & -x_{12} \\ -\overline{x_{12}} & x_{11} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} = (x_{22} x_{11} - x_{12} \overline{x_{12}}) 1_{\mathcal{E}}.$$

This is the desired formula for  $f = 1_{\mathcal{E}}$ . In the general case, note that  $x \in \mathcal{E}^+$  if and only if  $fx$  is symmetric with respect to  $^{-tr}$  and apply the above formula to  $fx$ .

(iv) The remaining two cases for  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$  are treated similarly, like (i) and (ii) with  $\mathcal{E}^+$  replaced by  $\mathcal{E}$ .  $\square$

The question immediately arises, whether the map  $\iota$  of Definition 5.1 is or can be extended to an equivalence of  $\text{Bil}_{\mathcal{A}}(\mathcal{V})$  onto  $\text{Bil}_{\mathcal{A}}(\mathcal{V}^*)$ . This is

clearly the case for two-dimensional  $\text{End}_{\mathcal{A}}(\mathcal{V})$ . It may fail for two-dimensional  $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  with four-dimensional commutative  $\text{End}_{\mathcal{A}}(\mathcal{V})$  for the simple reason that the nontrivial automorphism of the real quadratic subfield does not necessarily extend to the whole of  $\text{End}_{\mathcal{A}}(\mathcal{V})$ . For  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$  one gets a nice canonical answer, cf. Proposition 5.4 below. For  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2 \times 2}$  the answer is still positive, but the proof is computational and we omit it. Finally, for  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{H}^{2 \times 2}$  the map  $\iota$  no longer extends to an equivalence.

**PROPOSITION 5.4.** *Let  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$ . Then any nonzero  $\psi \in \text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  defines an equivalence  $\text{Bil}_{\mathcal{A}}(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}(\mathcal{V}^*) : \phi \mapsto \psi\phi\psi^{tr}$  which restricts to a map  $\iota : \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  with the properties described in Proposition 5.3.*

*Proof.* If  $\mathcal{V}$  is a simple  $\mathcal{A}$ -module, obviously any nonzero element of  $\text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  is invertible if viewed as an  $\mathcal{A}$ -homomorphism from  $\mathcal{V}^*$  to  $\mathcal{V}$ . Otherwise,  $\mathcal{V} \cong \mathcal{V}_0 \oplus \mathcal{V}_0$  for some simple  $\mathcal{A}$ -module  $\mathcal{V}_0$ . Any  $\mathcal{A}$ -isomorphism  $\mathcal{V}_0 \rightarrow \mathcal{V}_0^*$  gives rise to an invertible element of  $\text{Bil}_{\mathcal{A}}^-(\mathcal{V})$ , which therefore consists of 0 and invertible elements, since it is one-dimensional. One easily checks that any nonzero  $\psi \in \text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  leads to an equivalence, whose associated isomorphism  $\text{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*) \rightarrow \text{End}_{\mathcal{A}}(\mathcal{V}^* \oplus \mathcal{V})$  is induced by conjugation with  $\text{diag}(-\psi^{-1}, \psi)$ . Finally, for any  $\phi \in \text{Bil}_{\mathcal{A}}^+(\mathcal{V})$  one has  $\phi(\psi\phi\psi^{tr}) = q(\phi)\text{id}_{\mathcal{V}}$  with  $q(\phi) := n(\psi\phi)$ , where  $n$  is the reduced norm map of the quaternion algebra  $\text{End}_{\mathcal{A}}(\mathcal{V}^*)$ . This is so, since  $\phi(\psi\phi\psi^{tr}) = -(\phi\psi)^2$  and  $\phi\psi$  lies in  $\text{End}_{\mathcal{A}}(\mathcal{V}^*)$  and is of trace zero by  $\text{tr}(\phi\psi) = \text{tr}((\phi\psi)^{tr}) = \text{tr}(-\psi\phi) = -\text{tr}(\phi\psi)$ .  $\square$

The next result normalizes  $\iota$  and interprets it in the integral environment of  $\text{Bil}_{\Lambda}^+(L)$ .

**THEOREM 5.5.** *Let  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$  with  $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ .*

- (i) *There is a unique  $\text{Aut}(\text{Bil}_{\Lambda}(L))$ -invariant quadratic form  $q : \text{Bil}_{\Lambda}^+(L) \rightarrow \mathbf{Z}$  such that the  $\text{gcd}(q(\phi))$  for  $\phi \in \text{Bil}_{\Lambda}^+(L)$  is 1, and  $q(\phi) > 0$  for  $\phi \in \text{Bil}_{\Lambda}^+(L)$  positive definite.*
- (ii) *There is a unique constant  $c \in \mathbf{Z}$  satisfying  $\det(\phi) = cq(\phi)^m$  with  $m = 2^{-1} \dim_{\mathbf{Q}} \mathcal{V}$  for all  $\phi \in \text{Bil}_{\Lambda}^+(L)$ . (Clearly  $c \geq 1$ .)*
- (iii) *There is a unique  $\text{Aut}(\text{Bil}_{\Lambda}(L))$ -monomorphism  $\iota : \text{Bil}_{\Lambda}^+(L) \rightarrow \text{Bil}_{\Lambda}^+(L^*)$  mapping positive definite forms on positive definite ones such that the image of  $\iota$  is not contained in  $p \text{Bil}_{\Lambda}^+(L^*)$  for any integer  $p \geq 2$ .*

- (iv) *There is a unique constant  $c_0 \in \mathbf{Z}$  with  $\phi^\iota \phi = c_0 q(\phi) id_L$  for all  $\phi \in \text{Bil}_\Lambda^+(L)$ . Moreover  $c$  divides  $c_0^n$ , where  $n = \dim_{\mathbf{Q}} \mathcal{V}$ . (In fact  $\det(\phi^\iota) = c_0^n c^{-1} q(\phi)^m$  for all  $\phi \in \text{Bil}_\Lambda^+(L)$ .)*
- (v)  *$\text{Aut}(\text{Bil}_\Lambda^+(L)) \leq \text{O}(\text{Bil}_\Lambda^+(L), q)$  is a subgroup of finite index.*

*Proof.* Let  $\text{Bil}_\Lambda^+(L) = \langle \phi_1, \phi_2, \dots, \phi_d \rangle_{\mathbf{Z}}$  (with  $d = 3, 4$ , resp.  $6$  for  $K = \mathbf{R}, \mathbf{C}$ , resp.  $\mathbf{H}$ ). Choose the isomorphism  $\iota$  of Proposition 5.3 by multiplying with a suitable positive rational number such that  $\text{Bil}_\Lambda^+(L)$  is mapped into  $\text{Bil}_\Lambda^+(L^*)$  but not into a proper multiple of  $\text{Bil}_\Lambda^+(L^*)$ . After rescaling  $q$  of Proposition 5.3 appropriately, one gets a quadratic form  $\tilde{q} \in \mathbf{Z}[x_1, \dots, x_d]$  with

$$\left( \sum_{i=1}^d x_i \phi_i^\iota \right) \left( \sum_{i=1}^d x_i \phi_i \right) = \tilde{q}(x_1, \dots, x_d) id_L.$$

Since  $\mathbf{Z}[x_1, \dots, x_d]$  is a unique factorization domain, one obtains a constant  $c_0$  and a quadratic form  $q$  as required in (i) and (iv). Also by taking determinants, the unique factorization property yields  $\det(\phi) = cq(\phi)$  with a unique integer  $c$  dividing  $c_0^n$ . Since  $\det(g\phi) = \det(g)^2 \det(\phi) = \det(\phi)$  for  $g \in N(L)$ , one sees that  $q$  is  $\text{Aut}(\text{Bil}_\Lambda^+(L))$ -invariant, at least up to sign. And since the action respects positive definiteness, one gets invariance. One clearly has  $(g\phi)^\iota = g^{-tr} \phi^\iota$  for all  $g \in N(L)$  and all  $\phi \in \text{Bil}_\Lambda^+(L)$  of nonzero determinant. But since all other elements of  $\text{Bil}_\Lambda^+(L)$  are rational linear combinations of these, one obtains the equation for all  $\phi \in \text{Bil}_\Lambda^+(L)$ .

To prove (v) we first note that, by a standard Lie group argument, the group  $S$  of norm 1 units of  $\text{End}_{\mathcal{A}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V})$  is mapped onto the 1-component of  $\text{O}(\text{Bil}_{\mathbf{R} \otimes \Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$ . Also it is well known that the subgroup  $\Gamma$  of norm 1 elements of  $\text{End}_\Lambda(L)^*$  (which is clearly of finite index in  $N(L)$ ) has finite covolume in  $S$ . This implies that  $\text{Aut}(\text{Bil}_\Lambda^+(L))$  is of finite covolume in  $\text{O}(\text{Bil}_{\mathbf{R} \otimes \Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$  and therefore of finite index in  $\text{O}(\text{Bil}_\Lambda^+(L), q)$ .

It follows from (v) and the fact that the signature of  $q$  is  $(1, d-1)$  that  $\text{Aut}(\text{Bil}_\Lambda^+(L))$  acts absolutely irreducibly on  $\text{Bil}_\Lambda^+(L)$ . This again implies that the invariant quadratic form  $q$  is unique up to rational multiples, i. e. unique with the properties specified in (i). It also implies the uniqueness of  $\iota$  in (iii). The uniqueness of the constants  $c_0$  and  $c$  now follows from the considerations at the beginning of the proof.  $\square$

The corresponding results for the other examples given in Example 5.2 are left as exercises to the reader, who should note however that the action of  $\text{O}(\text{Bil}_\Lambda^+(L), q)$  on  $\text{Bil}_\Lambda^+(L)$  need not be absolutely irreducible any more.

The next topic is to set the concepts of this chapter into relation with modular lattices as introduced by Quebbemann in [Que95]; cf. also [SSch98] and [Ple98] for surveys.

DEFINITION 5.6.

(i)  $\phi \in \text{Bil}_\Lambda^+(L)$  is said to be  $k$ -modular, for  $k \in \mathbf{Z}$ , if  $(L^\sharp, k\phi)$  is isometric to  $(L, \phi)$ , where  $L^\sharp = \{l \in \mathcal{V} \mid \phi(l, L) \subseteq \mathbf{Z}\}$ . (Note the Gram matrix of  $\phi$  on  $L^\sharp$  is inverse to the Gram matrix on  $L$  if one chooses the bases dual to each other.)

(ii)  $\text{Bil}_\Lambda^+(L)$  is called *modular* if  $\text{Bil}_\Lambda^+(L)$  is special by the maps  $\iota: \text{Bil}_\Lambda^+(L) \rightarrow \text{Bil}_\Lambda(L^*)$  and  $q: \text{Bil}_\Lambda^+(L) \rightarrow \mathbf{Z}$ , cf. Definition 5.1, such that  $\iota$  is (the restriction to  $\text{Bil}_\Lambda^+(L)$  of) an induced equivalence; cf. Definition 4.3.

Clearly, if  $\text{Bil}_\Lambda^+(L)$  is modular, each nondegenerate  $\phi \in \text{Bil}_\Lambda^+(L)$  is  $c_0 q(\phi)$ -modular with  $c_0$  as in Theorem 5.5, and the isometries are all given by the same map. Some examples of two-dimensional modular lattices of covariant forms have already been investigated in the literature, cf. e. g. [Neb98b] where even the Hermite function was discussed for some examples or [Neb96a], where the extremal 3-modular lattice in dimension 24 was discovered. Here the main issue concerns the cases with  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$  or  $\mathbf{C}^{2 \times 2}$ , since  $\mathbf{H}^{2 \times 2}$  cannot occur. Example 6.6 (i) provides an example where  $\text{Bil}_\Lambda^+(L)$  is special without being modular. It should be emphasized that induced equivalence between  $\text{Bil}_\Lambda(L)$  and  $\text{Bil}_\Lambda(L^*)$  is not an uncommon phenomenon. For instance it occurs whenever  $L$  and  $L^*$  are  $\Lambda$ -isomorphic. That the induced equivalence is  $\iota$ , is rather rare.

PROPOSITION 5.7. *Let  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$  and assume  $\text{Bil}_\Lambda^-(L) = \mathbf{Z}\psi_1$  and  $\text{Bil}_\Lambda^-(L^*) = \mathbf{Z}\psi_2$  with  $\psi_1\psi_2 = e \cdot \text{id}_L$  for some natural number  $e$ .*

- (i) *If  $e = 1$  then  $\text{Bil}_\Lambda(L)$  is modular, with  $\iota$  induced by  $\psi_2$ .*
- (ii) *If  $\psi_1$  and  $\psi_2$  do not have the same elementary divisors, then  $\text{Bil}_\Lambda^+(L)$  is not modular.*
- (iii) *If  $e^{\dim(L)} \neq \det(\psi_2)^2$  then  $\text{Bil}_\Lambda^+(L)$  is not modular.*

*Proof.* (i) This follows along the lines of Proposition 5.4. That  $\text{Bil}_\Lambda(L)$  is mapped onto  $\text{Bil}_\Lambda(L^*)$  follows from the fact that  $\det(\psi_2) = \pm 1$ .

(ii) This is because induced equivalence respects elementary divisors.

(iii) This can be derived from (ii) by taking determinants. It can also be obtained from the observation that  $\psi_2$  induces  $e \cdot \iota$ .  $\square$

## EXAMPLE 5.8.

(i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the  $e$  of Proposition 5.7 is 1, 2, 3 and 6. In all cases  $\text{Bil}_\Lambda^+(L)$  is modular and  $c_0$  is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.

(ii) In Example 2.2 (ii), choose  $f_0$  to be  $m$ -modular for some natural number  $m$ . Then  $\text{Bil}_\Lambda^+(L \oplus L)$  (in the notation of Example 2.2 (ii)) is modular, where the  $e$  of Proposition 5.7 is equal to  $m$ , as is  $c_0$ .

To test whether  $\text{Bil}_\Lambda^+(L)$  is modular, one can simply compute the images of a  $\mathbf{Z}$ -basis of  $\text{Bil}_\Lambda^+(L)$  under  $\iota$  as described in Theorem 5.5 and find a simultaneous isometry of  $L$  to  $L^*$  (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PLS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with  $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2 \times 2}$ . One such example, involving the Leech lattice with  $\text{End}_{\mathcal{A}}(\mathcal{V})$  a non-split quaternion algebra over  $\mathbf{Q}[\sqrt{-7}]$ , is sketched in the last chapter of [Ple96].

## 6. SOME THREE-DIMENSIONAL LATTICES OF COVARIANT FORMS

This chapter is devoted to some examples in the case where  $\text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2 \times 2}$  and where the depth of  $\text{Bil}_\Lambda(L)$  is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of  $\text{Bil}_\Lambda^+(L)$  compare to the orthogonal group of  $(\text{Bil}_\Lambda^+(L), q)$ ? The simplest case is  $\text{End}_\Lambda(L) \cong \mathbf{Z}^{2 \times 2}$ , where all these questions can be answered.

**THEOREM 6.1.** *Let  $\text{End}_\Lambda(L) \cong \mathbf{Z}^{2 \times 2}$ . Then  $L = L_0 \oplus L_0$  for some irreducible  $\Lambda$ -lattice  $L_0$ . Let  $\phi_0$  be the positive definite generator of  $\text{Bil}_\Lambda^+(L_0)$ . Then  $c$ ,  $c_0$ , and  $q$ , introduced in Theorem 5.5, are as follows.*

- (i) *With respect to a suitable basis of  $\text{Bil}_\Lambda^+(L)$ , the quadratic form  $q$  of Theorem 5.5 becomes  $xy - z^2$ .*
- (ii)  $c = \det(\phi_0)^2$ .
- (iii)  $c_0$  *is the exponent of  $L_0^\sharp/L_0$ , i. e. the biggest elementary divisor of a Gram matrix of  $\phi_0$ .*