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## 5. INVERSION AND MODULARITY

Since  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}) \subseteq \operatorname{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*)$ , the inverse  $\phi^{-1}$  of a nondegenerate  $\phi \in \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is well defined and lies in  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ . By Cramer's rule inversion is a rational map from  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  to  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ , more precisely there is a homogeneous polynomial map  $P \colon \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}) \to \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  such that  $\phi^P \phi = \det(\phi) \cdot id_{\mathcal{V}}$ . Viewing this as an identity of matrices with polynomial entries, one might cancel out the greatest common divisor of all occurring entries and get new polynomial maps  $p \colon \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}) \to \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  and  $d \colon \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}) \to \mathbf{Q}$  with  $\phi^P \phi = d(\phi) \cdot id_{\mathcal{V}}$ . The properties of the map p have not been studied in this generality. The aim here is to investigate the simplest case, where p is homogeneous of degree 1, i.e. a  $\mathbf{Q}$ -linear map  $\iota$ , as it is called in the sequel. Of course, the same analysis can be done with  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V})$ . The question whether such a  $\iota$  is an equivalence, will be treated later in this section.

DEFINITION 5.1. Let R be one of  $\mathbf{Z}$  or  $\mathbf{Q}$ . Then  $\mathrm{Bil}_{\Lambda_R}(L_R)$  is called *special* if there is an R-linear map  $\iota \colon \mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(L_R^*)$  and a quadratic form  $q \colon \mathrm{Bil}_{\Lambda_R}(L_R) \to R$  such that for any nondegenerate  $\phi \in \mathrm{Bil}_{\Lambda_R}(L_R)$  one has  $\phi^\iota \phi = q(\phi) \, id_{L_R}$ . Analogous definitions hold for  $\mathrm{Bil}_{\Lambda_R}^+(L_R)$ 

## EXAMPLE 5.2.

- (i) One-dimensional lattices of covariant forms are special for trivial reasons.
- (ii) If  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V})$  is two-dimensional, then it is special. This is because  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V})$  can be viewed as a free  $Z(\mathcal{A})$ -module and for two-dimensional algebras  $\mathcal{B}$  one has a canonical automorphism  $\kappa$  of  $\mathcal{B}$  such that  $b^{\kappa} = n(b)b^{-1}$  for all  $b \in \mathcal{B}^*$ , where  $n \colon \mathcal{B} \to F$  is the norm map with respect to the regular representation. (Note that  $Z(\mathcal{A}) = \operatorname{End}_{\mathcal{A}}(\mathcal{V})$  in the present situation.)
- (iii) If  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is two-dimensional then it is special. This is because  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  can be viewed as a free  $Z(\mathcal{A})^+$ -module, where

$$Z(\mathcal{A})^+ := \{ \varphi \in Z(\mathcal{A}) \mid \varphi^\circ = \varphi \}.$$

Here are some more interesting examples.

PROPOSITION 5.3. Let  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$  with  $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ . Then  $\operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  is special. In the first two cases also  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V})$  is special.

- (i) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{R}^{2 \times 2}$ . Then  $\mathcal{E}$  is a quaternion algebra over  $\mathbf{Q}$ . Denote its canonical involution by  $\omega'$  and its reduced norm by n. Clearly, n is a quadratic form and  $\omega'(\phi)$   $\phi = n(\phi) 1$  holds for all elements  $\phi \in \mathcal{E}$ . With  $\iota := \omega'_{|\mathcal{E}^+|}$  and  $q := n_{|\mathcal{E}^+|}$  one gets the desired formula.
- (ii) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{C}^{2 \times 2}$ . Then  $\mathcal{E}$  is a quaternion algebra over the imaginary quadratic number field  $Z := Z(\mathcal{A})$ . Denote its canonical involution by  $\omega'$  and its reduced norm by n. The involution 'induces the nontrivial Galois automorphism of  $(Z/\mathbf{Q})$ , and therefore one checks quite easily, using [Scha85] Theorem 11.2 (ii) of Chapter 8, that the norm n maps  $\mathcal{E}^+$  into  $\mathbf{Q}$ . Now one argues as in (i).
- (iii) Let  $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{H}^{2\times 2}$ . Then  $\mathcal{E} \cong D^{2\times 2}$ , where D is a positive definite quaternion algebra over  $\mathbf{Q}$  (with canonical involution  $\omega'$ ). Indeed,  $\mathcal{E}$  carries an involution of the first kind and hence cannot be of index 4. Since ' is a positive involution one sees from the proof of Theorem 13.3 of Chapter 8 in [Scha85] that  $x^{\bullet} = f^{-1}\overline{x}^{tr}f$  for all  $x \in \mathcal{E}$ , where  $f = \overline{f}^{tr} \in \mathcal{E}^*$  and  $\overline{(x_{ij})} = (\overline{x_{ij}})$  for all  $(x_{ij}) \in D^{2\times 2} \equiv \mathcal{E}$ . If  $(x_{ij}) \in \mathcal{E}$  is symmetric with respect to  $^{-tr}$  one checks

$$(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix}$$
 with  $\overline{x_{ii}} = x_{ii}$  for  $i = 1, 2$ 

and 
$$\begin{pmatrix} x_{22} & -x_{12} \\ -\overline{x_{12}} & x_{11} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} = (x_{22} x_{11} - x_{12} \overline{x_{12}}) 1_{\mathcal{E}}.$$

This is the desired formula for  $f = 1_{\mathcal{E}}$ . In the general case, note that  $x \in \mathcal{E}^+$  if and only if fx is symmetric with respect to fx and apply the above formula to fx.

(iv) The remaining two cases for  $Bil_{\mathcal{A}}(\mathcal{V})$  are treated similarly, like (i) and (ii) with  $\mathcal{E}^+$  replaced by  $\mathcal{E}$ .  $\square$ 

The question immediately arises, whether the map  $\iota$  of Definition 5.1 is or can be extended to an equivalence of  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V})$  onto  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V}^*)$ . This is

clearly the case for two-dimensional  $\operatorname{End}_{\mathcal{A}}(\mathcal{V})$ . It may fail for two-dimensional  $\operatorname{Bil}^+_{\mathcal{A}}(\mathcal{V})$  with four-dimensional commutative  $\operatorname{End}_{\mathcal{A}}(\mathcal{V})$  for the simple reason that the nontrivial automorphism of the real quadratic subfield does not necessarily extend to the whole of  $\operatorname{End}_{\mathcal{A}}(\mathcal{V})$ . For  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$  one gets a nice canonical answer, cf. Proposition 5.4 below. For  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2 \times 2}$  the answer is still positive, but the proof is computational and we omit it. Finally, for  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{H}^{2 \times 2}$  the map  $\iota$  no longer extends to an equivalence.

PROPOSITION 5.4. Let  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$ . Then any nonzero  $\psi \in \operatorname{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  defines an equivalence  $\operatorname{Bil}_{\mathcal{A}}(\mathcal{V}) \to \operatorname{Bil}_{\mathcal{A}}(\mathcal{V}^*)$ :  $\phi \mapsto \psi \phi \psi^{tr}$  which restricts to a map  $\iota \colon \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}) \to \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$  with the properties described in Proposition 5.3.

*Proof.* If  $\mathcal{V}$  is a simple  $\mathcal{A}$ -module, obviously any nonzero element of  $\operatorname{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  is invertible if viewed as an  $\mathcal{A}$ -homomorphism from  $\mathcal{V}^*$  to  $\mathcal{V}$ . Otherwise,  $\mathcal{V} \cong \mathcal{V}_0 \oplus \mathcal{V}_0$  for some simple  $\mathcal{A}$ -module  $\mathcal{V}_0$ . Any  $\mathcal{A}$ -isomorphism  $\mathcal{V}_0 \to \mathcal{V}_0^*$  gives rise to an invertible element of  $\operatorname{Bil}_{\mathcal{A}}^-(\mathcal{V})$ , which therefore consists of 0 and invertible elements, since it is one-dimensional. One easily checks that any nonzero  $\psi \in \operatorname{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$  leads to an equivalence, whose associated isomorphism  $\operatorname{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*) \to \operatorname{End}_{\mathcal{A}}(\mathcal{V}^* \oplus \mathcal{V})$  is induced by conjugation with  $\operatorname{diag}(-\psi^{-1},\psi)$ . Finally, for any  $\phi \in \operatorname{Bil}_{\mathcal{A}}^+(\mathcal{V})$  one has  $\phi(\psi\phi\psi^{tr}) = q(\phi)id_{\mathcal{V}}$  with  $q(\phi) := n(\psi\phi)$ , where n is the reduced norm map of the quaternion algebra  $\operatorname{End}_{\mathcal{A}}(\mathcal{V}^*)$ . This is so, since  $\phi(\psi\phi\psi^{tr}) = -(\phi\psi)^2$  and  $\phi\psi$  lies in  $\operatorname{End}_{\mathcal{A}}(\mathcal{V}^*)$  and is of trace zero by  $tr(\phi\psi) = tr((\phi\psi)^{tr}) = tr(-\psi\phi) = -tr(\phi\psi)$ .  $\square$ 

The next result normalizes  $\iota$  and interprets it in the integral environment of  $\operatorname{Bil}_{\Lambda}^+(L)$ .

THEOREM 5.5. Let  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$  with  $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ .

- (i) There is a unique  $\operatorname{Aut}(\operatorname{Bil}_{\Lambda}(L))$ -invariant quadratic form  $q \colon \operatorname{Bil}_{\Lambda}^+(L) \to \mathbf{Z}$  such that the  $\gcd(q(\phi))$  for  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$  is 1, and  $q(\phi) > 0$  for  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$  positive definite.
- (ii) There is a unique constant  $c \in \mathbf{Z}$  satisfying  $\det(\phi) = cq(\phi)^m$  with  $m = 2^{-1} \dim_{\mathbf{Q}} \mathcal{V}$  for all  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$ . (Clearly  $c \geq 1$ .)
- (iii) There is a unique  $\operatorname{Aut}(\operatorname{Bil}_{\Lambda}(L))$ -monomorphism  $\iota \colon \operatorname{Bil}_{\Lambda}^+(L) \to \operatorname{Bil}_{\Lambda}^+(L^*)$  mapping positive definite forms on positive definite ones such that the image of  $\iota$  is not contained in  $p\operatorname{Bil}_{\Lambda}^+(L^*)$  for any integer  $p \geq 2$ .

- (iv) There is a unique constant  $c_0 \in \mathbf{Z}$  with  $\phi^{\iota}\phi = c_0q(\phi)id_L$  for all  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$ . Moreover c divides  $c_0^n$ , where  $n = \dim_{\mathbf{Q}} \mathcal{V}$ . (In fact  $\det(\phi^{\iota}) = c_0^n c^{-1} q(\phi)^m$  for all  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$ .)
- (v)  $\operatorname{Aut}(\operatorname{Bil}_{\Lambda}^+(L)) \leq \operatorname{O}(\operatorname{Bil}_{\Lambda}^+(L), q)$  is a subgroup of finite index.

*Proof.* Let  $\operatorname{Bil}_{\Lambda}^+(L) = \langle \phi_1, \phi_2, \dots, \phi_d \rangle_{\mathbf{Z}}$  (with d = 3, 4, resp. 6 for  $K = \mathbf{R}$ ,  $\mathbf{C}$ , resp.  $\mathbf{H}$ ). Choose the isomorphism  $\iota$  of Proposition 5.3 by multiplying with a suitable positive rational number such that  $\operatorname{Bil}_{\Lambda}^+(L)$  is mapped into  $\operatorname{Bil}_{\Lambda}^+(L^*)$  but not into a proper multiple of  $\operatorname{Bil}_{\Lambda}^+(L^*)$ . After rescaling q of Proposition 5.3 appropriately, one gets a quadratic form  $\widetilde{q} \in \mathbf{Z}[x_1, \dots, x_d]$  with

$$\left(\sum_{i=1}^d x_i \phi_i^{\iota}\right) \left(\sum_{i=1}^d x_i \phi_i\right) = \widetilde{q}(x_1, \ldots, x_d) i d_L.$$

Since  $\mathbf{Z}[x_1,\ldots,x_d]$  is a unique factorization domain, one obtains a constant  $c_0$  and a quadratic form q as required in (i) and (iv). Also by taking determinants, the unique factorization property yields  $\det(\phi) = cq(\phi)$  with a unique integer c dividing  $c_0^n$ . Since  $\det(g\phi) = \det(g)^2 \det(\phi) = \det(\phi)$  for  $g \in N(L)$ , one sees that q is  $\operatorname{Aut}(\operatorname{Bil}_{\Lambda}^+(L))$ -invariant, at least up to sign. And since the action respects positive definiteness, one gets invariance. One clearly has  $(g\phi)^{\iota} = g^{-tr}\phi^{\iota}$  for all  $g \in N(L)$  and all  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$  of nonzero determinant. But since all other elements of  $\operatorname{Bil}_{\Lambda}^+(L)$  are rational linear combinations of these, one obtains the equation for all  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$ .

To prove (v) we first note that, by a standard Lie group argument, the group S of norm 1 units of  $\operatorname{End}_{\mathcal{A}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V})$  is mapped onto the 1-component of  $\operatorname{O}(\operatorname{Bil}^+_{\mathbf{R} \otimes_{\mathbf{A}}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$ . Also it is well known that the subgroup  $\Gamma$  of norm 1 elements of  $\operatorname{End}_{\Lambda}(L)^*$  (which is clearly of finite index in N(L)) has finite covolume in S. This implies that  $\operatorname{Aut}(\operatorname{Bil}^+_{\Lambda}(L))$  is of finite covolume in  $\operatorname{O}(\operatorname{Bil}^+_{\Lambda}(L), q)$ .

It follows from (v) and the fact that the signature of q is (1, d-1) that  $\operatorname{Aut}(\operatorname{Bil}^+_{\Lambda}(L))$  acts absolutely irreducibly on  $\operatorname{Bil}^+_{\Lambda}(L)$ . This again implies that the invariant quadratic form q is unique up to rational multiples, i.e. unique with the properties specified in (i). It also implies the uniqueness of  $\iota$  in (iii). The uniqueness of the constants  $c_0$  and c now follows from the considerations at the beginning of the proof.  $\square$ 

The corresponding results for the other examples given in Example 5.2 are left as exercises to the reader, who should note however that the action of  $O(\operatorname{Bil}^+_{\Lambda}(L), q)$  on  $\operatorname{Bil}^+_{\Lambda}(L)$  need not be absolutely irreducible any more.

The next topic it to set the concepts of this chapter into relation with modular lattices as introduced by Quebbemann in [Que95]; cf. also [SSch98] and [Ple98] for surveys.

### DEFINITION 5.6.

- (i)  $\phi \in \operatorname{Bil}_{\Lambda}^+(L)$  is said to be k-modular, for  $k \in \mathbb{Z}$ , if  $(L^{\sharp}, k\phi)$  is isometric to  $(L, \phi)$ , where  $L^{\sharp} = \{l \in \mathcal{V} \mid \phi(l, L) \subseteq \mathbb{Z}\}$ . (Note the Gram matrix of  $\phi$  on  $L^{\sharp}$  is inverse to the Gram matrix on L if one chooses the bases dual to each other.)
- (ii)  $\operatorname{Bil}_{\Lambda}^+(L)$  is called *modular* if  $\operatorname{Bil}_{\Lambda}^+(L)$  is special by the maps  $\iota \colon \operatorname{Bil}_{\Lambda}^+(L) \to \operatorname{Bil}_{\Lambda}(L^*)$  and  $q \colon \operatorname{Bil}_{\Lambda}^+(L) \to \mathbf{Z}$ , cf. Definition 5.1, such that  $\iota$  is (the restriction to  $\operatorname{Bil}_{\Lambda}^+(L)$  of) an induced equivalence; cf. Definition 4.3.

Clearly, if  $\operatorname{Bil}^+_{\Lambda}(L)$  is modular, each nondegenerate  $\phi \in \operatorname{Bil}^+_{\Lambda}(L)$  is  $c_0q(\phi)$ -modular with  $c_0$  as in Theorem 5.5, and the isometries are all given by the same map. Some examples of two-dimensional modular lattices of covariant forms have already been investigated in the literature, cf. e. g. [Neb98b] where even the Hermite function was discussed for some examples or [Neb96a], where the extremal 3-modular lattice in dimension 24 was discovered. Here the main issue concerns the cases with  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2\times 2}$  or  $\mathbf{C}^{2\times 2}$ , since  $\mathbf{H}^{2\times 2}$  cannot occur. Example 6.6 (i) provides an example where  $\operatorname{Bil}^+_{\Lambda}(L)$  is special without being modular. It should be emphasized that induced equivalence between  $\operatorname{Bil}_{\Lambda}(L)$  and  $\operatorname{Bil}_{\Lambda}(L^*)$  is not an uncommon phenomenon. For instance it occurs whenever L and  $L^*$  are  $\Lambda$ -isomorphic. That the induced equivalence is  $\iota$ , is rather rare.

PROPOSITION 5.7. Let  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$  and assume  $\operatorname{Bil}_{\Lambda}^{-}(L) = \mathbf{Z}\psi_{1}$  and  $\operatorname{Bil}_{\Lambda}^{-}(L^{*}) = \mathbf{Z}\psi_{2}$  with  $\psi_{1}\psi_{2} = e \cdot id_{L}$  for some natural number e.

- (i) If e = 1 then  $Bil_{\Lambda}(L)$  is modular, with  $\iota$  induced by  $\psi_2$ .
- (ii) If  $\psi_1$  and  $\psi_2$  do not have the same elementary divisors, then  $\operatorname{Bil}^+_{\Lambda}(L)$  is not modular.
- (iii) If  $e^{\dim(L)} \neq \det(\psi_2)^2$  then  $\operatorname{Bil}_{\Lambda}^+(L)$  is not modular.

*Proof.* (i) This follows along the lines of Proposition 5.4. That  $\operatorname{Bil}_{\Lambda}(L)$  is mapped onto  $\operatorname{Bil}_{\Lambda}(L^*)$  follows from the fact that  $\det(\psi_2)=\pm 1$ .

- (ii) This is because induced equivalence respects elementary divisors.
- (iii) This can be derived from (ii) by taking determinants. It can also be obtained from the observation that  $\psi_2$  induces  $e \cdot \iota$ .

### EXAMPLE 5.8.

- (i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the e of Proposition 5.7 is 1, 2, 3 and 6. In all cases  $\operatorname{Bil}_{\Lambda}^+(L)$  is modular and  $c_0$  is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.
- (ii) In Example 2.2 (ii), choose  $f_0$  to be m-modular for some natural number m. Then  $\operatorname{Bil}^+_{\Lambda}(L \oplus L)$  (in the notation of Example 2.2 (ii)) is modular, where the e of Proposition 5.7 is equal to m, as is  $c_0$ .

To test whether  $\operatorname{Bil}_{\Lambda}^+(L)$  is modular, one can simply compute the images of a  $\mathbf{Z}$ -basis of  $\operatorname{Bil}_{\Lambda}^+(L)$  under  $\iota$  as described in Theorem 5.5 and find a simultaneous isometry of L to  $L^*$  (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PIS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with  $\mathbf{R} \otimes_{\mathbf{Q}} \operatorname{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2\times 2}$ . One such example, involving the Leech lattice with  $\operatorname{End}_{\mathcal{A}}(\mathcal{V})$  a non-split quaternion algebra over  $\mathbf{Q}[\sqrt{-7}]$ , is sketched in the last chapter of [Ple96].

# 6. Some three-dimensional lattices of covariant forms

This chapter is devoted to some examples in the case where  $\operatorname{End}_{\mathcal{A}}(\mathcal{V})\cong \mathbf{Q}^{2\times 2}$  and where the depth of  $\operatorname{Bil}_{\Lambda}(L)$  is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of  $\operatorname{Bil}_{\Lambda}^+(L)$  compare to the orthogonal group of  $(\operatorname{Bil}_{\Lambda}^+(L), q)$ ? The simplest case is  $\operatorname{End}_{\Lambda}(L) \cong \mathbf{Z}^{2\times 2}$ , where all these questions can be answered.

THEOREM 6.1. Let  $\operatorname{End}_{\Lambda}(L) \cong \mathbf{Z}^{2\times 2}$ . Then  $L = L_0 \oplus L_0$  for some irreducible  $\Lambda$ -lattice  $L_0$ . Let  $\phi_0$  be the positive definite generator of  $\operatorname{Bil}_{\Lambda}^+(L_0)$ . Then c,  $c_0$ , and q, introduced in Theorem 5.5, are as follows.

- (i) With respect to a suitable basis of  $\operatorname{Bil}_{\Lambda}^+(L)$ , the quadratic form q of Theorem 5.5 becomes  $xy-z^2$ .
- (ii)  $c = \det(\phi_0)^2$ .
- (iii)  $c_0$  is the exponent of  $L_0^{\sharp}/L_0$ , i. e. the biggest elementary divisor of a Gram matrix of  $\phi_0$ .