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Autor: Plesken, Wilhelm
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5. INVERSION AND MODULARITY

Since $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \subseteq \text{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$, the inverse ϕ^{-1} of a nondegenerate $\phi \in \text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ is well defined and lies in $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$. By Cramer's rule inversion is a rational map from $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ to $\text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$, more precisely there is a homogeneous polynomial map $P: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ such that $\phi^P \phi = \det(\phi) \cdot \text{id}_{\mathcal{V}}$. Viewing this as an identity of matrices with polynomial entries, one might cancel out the greatest common divisor of all occurring entries and get new polynomial maps $p: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ and $d: \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \mathbf{Q}$ with $\phi^p \phi = d(\phi) \cdot \text{id}_{\mathcal{V}}$. The properties of the map p have not been studied in this generality. The aim here is to investigate the simplest case, where p is homogeneous of degree 1, i.e. a \mathbf{Q} -linear map ι , as it is called in the sequel. Of course, the same analysis can be done with $\text{Bil}_{\mathcal{A}}(\mathcal{V})$. The question whether such a ι is an equivalence, will be treated later in this section.

DEFINITION 5.1. Let R be one of \mathbf{Z} or \mathbf{Q} . Then $\text{Bil}_{\Lambda_R}(L_R)$ is called *special* if there is an R -linear map $\iota: \text{Bil}_{\Lambda_R}(L_R) \rightarrow \text{Bil}_{\Lambda_R}(L_R^*)$ and a quadratic form $q: \text{Bil}_{\Lambda_R}(L_R) \rightarrow R$ such that for any nondegenerate $\phi \in \text{Bil}_{\Lambda_R}(L_R)$ one has $\phi^\iota \phi = q(\phi) \text{id}_{L_R}$. Analogous definitions hold for $\text{Bil}_{\Lambda_R}^+(L_R)$.

EXAMPLE 5.2.

(i) One-dimensional lattices of covariant forms are special for trivial reasons.

(ii) If $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ is two-dimensional, then it is special. This is because $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})$ -module and for two-dimensional algebras \mathcal{B} one has a canonical automorphism κ of \mathcal{B} such that $b^\kappa = n(b)b^{-1}$ for all $b \in \mathcal{B}^*$, where $n: \mathcal{B} \rightarrow F$ is the norm map with respect to the regular representation. (Note that $Z(\mathcal{A}) = \text{End}_{\mathcal{A}}(\mathcal{V})$ in the present situation.)

(iii) If $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ is two-dimensional then it is special. This is because $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})^+$ -module, where

$$Z(\mathcal{A})^+ := \{\varphi \in Z(\mathcal{A}) \mid \varphi^\circ = \varphi\}.$$

Here are some more interesting examples.

PROPOSITION 5.3. Let $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$ with $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Then $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ is special. In the first two cases also $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ is special.

Proof. Define $\mathcal{E} := \text{End}_{\mathcal{A}}(\mathcal{V}) \cong (e\mathcal{A}e)^{k \times k}$, where $e = e^\circ$ is a primitive $^\circ$ -invariant idempotent of \mathcal{A} and k is defined by $\mathcal{V} \cong (e\mathcal{A})^k$. In particular, the positive involution $^\circ$ on \mathcal{A} induces a positive involution $^\bullet$ on \mathcal{E} , $(a_{ij})^\bullet := (a_{ij}^\circ)^{tr}$, such that $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ can be identified with the subspace \mathcal{E}^+ of the symmetric elements in the algebra (\mathcal{E}, \bullet) with involution. It suffices to prove that there exists a \mathbf{Q} -vector space automorphism of \mathcal{E}^+ , also denoted by ι , and a \mathbf{Q} -valued quadratic form on \mathcal{E}^+ , also denoted by q , such that $\phi^\iota \phi = q(\phi) 1_{\mathcal{E}}$.

(i) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{R}^{2 \times 2}$. Then \mathcal{E} is a quaternion algebra over \mathbf{Q} . Denote its canonical involution by ω' and its reduced norm by n . Clearly, n is a quadratic form and $\omega'(\phi) \phi = n(\phi) 1$ holds for all elements $\phi \in \mathcal{E}$. With $\iota := \omega'|_{\mathcal{E}^+}$ and $q := n|_{\mathcal{E}^+}$ one gets the desired formula.

(ii) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{C}^{2 \times 2}$. Then \mathcal{E} is a quaternion algebra over the imaginary quadratic number field $Z := Z(\mathcal{A})$. Denote its canonical involution by ω' and its reduced norm by n . The involution $^\bullet$ induces the nontrivial Galois automorphism of (Z/\mathbf{Q}) , and therefore one checks quite easily, using [Scha85] Theorem 11.2 (ii) of Chapter 8, that the norm n maps \mathcal{E}^+ into \mathbf{Q} . Now one argues as in (i).

(iii) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{H}^{2 \times 2}$. Then $\mathcal{E} \cong D^{2 \times 2}$, where D is a positive definite quaternion algebra over \mathbf{Q} (with canonical involution ω'). Indeed, \mathcal{E} carries an involution of the first kind and hence cannot be of index 4. Since $^\bullet$ is a positive involution one sees from the proof of Theorem 13.3 of Chapter 8 in [Scha85] that $x^\bullet = f^{-1} \bar{x}^{tr} f$ for all $x \in \mathcal{E}$, where $f = \bar{f}^{tr} \in \mathcal{E}^*$ and $\overline{(x_{ij})} = (\overline{x_{ij}})$ for all $(x_{ij}) \in D^{2 \times 2} \cong \mathcal{E}$. If $(x_{ij}) \in \mathcal{E}$ is symmetric with respect to $^{-tr}$ one checks

$$(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} \quad \text{with } \overline{x_{ii}} = x_{ii} \text{ for } i = 1, 2$$

$$\text{and} \quad \begin{pmatrix} x_{22} & -x_{12} \\ -\overline{x_{12}} & x_{11} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} = (x_{22} x_{11} - x_{12} \overline{x_{12}}) 1_{\mathcal{E}}.$$

This is the desired formula for $f = 1_{\mathcal{E}}$. In the general case, note that $x \in \mathcal{E}^+$ if and only if fx is symmetric with respect to $^{-tr}$ and apply the above formula to fx .

(iv) The remaining two cases for $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ are treated similarly, like (i) and (ii) with \mathcal{E}^+ replaced by \mathcal{E} . \square

The question immediately arises, whether the map ι of Definition 5.1 is or can be extended to an equivalence of $\text{Bil}_{\mathcal{A}}(\mathcal{V})$ onto $\text{Bil}_{\mathcal{A}}(\mathcal{V}^*)$. This is

clearly the case for two-dimensional $\text{End}_{\mathcal{A}}(\mathcal{V})$. It may fail for two-dimensional $\text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ with four-dimensional commutative $\text{End}_{\mathcal{A}}(\mathcal{V})$ for the simple reason that the nontrivial automorphism of the real quadratic subfield does not necessarily extend to the whole of $\text{End}_{\mathcal{A}}(\mathcal{V})$. For $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$ one gets a nice canonical answer, cf. Proposition 5.4 below. For $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2 \times 2}$ the answer is still positive, but the proof is computational and we omit it. Finally, for $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{H}^{2 \times 2}$ the map ι no longer extends to an equivalence.

PROPOSITION 5.4. *Let $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$. Then any nonzero $\psi \in \text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$ defines an equivalence $\text{Bil}_{\mathcal{A}}(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}(\mathcal{V}^*) : \phi \mapsto \psi\phi\psi^{tr}$ which restricts to a map $\iota : \text{Bil}_{\mathcal{A}}^+(\mathcal{V}) \rightarrow \text{Bil}_{\mathcal{A}}^+(\mathcal{V}^*)$ with the properties described in Proposition 5.3.*

Proof. If \mathcal{V} is a simple \mathcal{A} -module, obviously any nonzero element of $\text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$ is invertible if viewed as an \mathcal{A} -homomorphism from \mathcal{V}^* to \mathcal{V} . Otherwise, $\mathcal{V} \cong \mathcal{V}_0 \oplus \mathcal{V}_0$ for some simple \mathcal{A} -module \mathcal{V}_0 . Any \mathcal{A} -isomorphism $\mathcal{V}_0 \rightarrow \mathcal{V}_0^*$ gives rise to an invertible element of $\text{Bil}_{\mathcal{A}}^-(\mathcal{V})$, which therefore consists of 0 and invertible elements, since it is one-dimensional. One easily checks that any nonzero $\psi \in \text{Bil}_{\mathcal{A}}^-(\mathcal{V}^*)$ leads to an equivalence, whose associated isomorphism $\text{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*) \rightarrow \text{End}_{\mathcal{A}}(\mathcal{V}^* \oplus \mathcal{V})$ is induced by conjugation with $\text{diag}(-\psi^{-1}, \psi)$. Finally, for any $\phi \in \text{Bil}_{\mathcal{A}}^+(\mathcal{V})$ one has $\phi(\psi\phi\psi^{tr}) = q(\phi)\text{id}_{\mathcal{V}}$ with $q(\phi) := n(\psi\phi)$, where n is the reduced norm map of the quaternion algebra $\text{End}_{\mathcal{A}}(\mathcal{V}^*)$. This is so, since $\phi(\psi\phi\psi^{tr}) = -(\phi\psi)^2$ and $\phi\psi$ lies in $\text{End}_{\mathcal{A}}(\mathcal{V}^*)$ and is of trace zero by $\text{tr}(\phi\psi) = \text{tr}((\phi\psi)^{tr}) = \text{tr}(-\psi\phi) = -\text{tr}(\phi\psi)$. \square

The next result normalizes ι and interprets it in the integral environment of $\text{Bil}_{\Lambda}^+(L)$.

THEOREM 5.5. *Let $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2 \times 2}$ with $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$.*

- (i) *There is a unique $\text{Aut}(\text{Bil}_{\Lambda}(L))$ -invariant quadratic form $q : \text{Bil}_{\Lambda}^+(L) \rightarrow \mathbf{Z}$ such that the $\text{gcd}(q(\phi))$ for $\phi \in \text{Bil}_{\Lambda}^+(L)$ is 1, and $q(\phi) > 0$ for $\phi \in \text{Bil}_{\Lambda}^+(L)$ positive definite.*
- (ii) *There is a unique constant $c \in \mathbf{Z}$ satisfying $\det(\phi) = cq(\phi)^m$ with $m = 2^{-1} \dim_{\mathbf{Q}} \mathcal{V}$ for all $\phi \in \text{Bil}_{\Lambda}^+(L)$. (Clearly $c \geq 1$.)*
- (iii) *There is a unique $\text{Aut}(\text{Bil}_{\Lambda}(L))$ -monomorphism $\iota : \text{Bil}_{\Lambda}^+(L) \rightarrow \text{Bil}_{\Lambda}^+(L^*)$ mapping positive definite forms on positive definite ones such that the image of ι is not contained in $p \text{Bil}_{\Lambda}^+(L^*)$ for any integer $p \geq 2$.*

- (iv) *There is a unique constant $c_0 \in \mathbf{Z}$ with $\phi^\iota \phi = c_0 q(\phi) id_L$ for all $\phi \in \text{Bil}_\Lambda^+(L)$. Moreover c divides c_0^n , where $n = \dim_{\mathbf{Q}} \mathcal{V}$. (In fact $\det(\phi^\iota) = c_0^n c^{-1} q(\phi)^m$ for all $\phi \in \text{Bil}_\Lambda^+(L)$.)*
- (v) *$\text{Aut}(\text{Bil}_\Lambda^+(L)) \leq \text{O}(\text{Bil}_\Lambda^+(L), q)$ is a subgroup of finite index.*

Proof. Let $\text{Bil}_\Lambda^+(L) = \langle \phi_1, \phi_2, \dots, \phi_d \rangle_{\mathbf{Z}}$ (with $d = 3, 4$, resp. 6 for $K = \mathbf{R}, \mathbf{C}$, resp. \mathbf{H}). Choose the isomorphism ι of Proposition 5.3 by multiplying with a suitable positive rational number such that $\text{Bil}_\Lambda^+(L)$ is mapped into $\text{Bil}_\Lambda^+(L^*)$ but not into a proper multiple of $\text{Bil}_\Lambda^+(L^*)$. After rescaling q of Proposition 5.3 appropriately, one gets a quadratic form $\tilde{q} \in \mathbf{Z}[x_1, \dots, x_d]$ with

$$\left(\sum_{i=1}^d x_i \phi_i^\iota \right) \left(\sum_{i=1}^d x_i \phi_i \right) = \tilde{q}(x_1, \dots, x_d) id_L.$$

Since $\mathbf{Z}[x_1, \dots, x_d]$ is a unique factorization domain, one obtains a constant c_0 and a quadratic form q as required in (i) and (iv). Also by taking determinants, the unique factorization property yields $\det(\phi) = cq(\phi)$ with a unique integer c dividing c_0^n . Since $\det(g\phi) = \det(g)^2 \det(\phi) = \det(\phi)$ for $g \in N(L)$, one sees that q is $\text{Aut}(\text{Bil}_\Lambda^+(L))$ -invariant, at least up to sign. And since the action respects positive definiteness, one gets invariance. One clearly has $(g\phi)^\iota = g^{-tr} \phi^\iota$ for all $g \in N(L)$ and all $\phi \in \text{Bil}_\Lambda^+(L)$ of nonzero determinant. But since all other elements of $\text{Bil}_\Lambda^+(L)$ are rational linear combinations of these, one obtains the equation for all $\phi \in \text{Bil}_\Lambda^+(L)$.

To prove (v) we first note that, by a standard Lie group argument, the group S of norm 1 units of $\text{End}_{\mathcal{A}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V})$ is mapped onto the 1-component of $\text{O}(\text{Bil}_{\mathbf{R} \otimes \Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$. Also it is well known that the subgroup Γ of norm 1 elements of $\text{End}_\Lambda(L)^*$ (which is clearly of finite index in $N(L)$) has finite covolume in S . This implies that $\text{Aut}(\text{Bil}_\Lambda^+(L))$ is of finite covolume in $\text{O}(\text{Bil}_{\mathbf{R} \otimes \Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$ and therefore of finite index in $\text{O}(\text{Bil}_\Lambda^+(L), q)$.

It follows from (v) and the fact that the signature of q is $(1, d-1)$ that $\text{Aut}(\text{Bil}_\Lambda^+(L))$ acts absolutely irreducibly on $\text{Bil}_\Lambda^+(L)$. This again implies that the invariant quadratic form q is unique up to rational multiples, i. e. unique with the properties specified in (i). It also implies the uniqueness of ι in (iii). The uniqueness of the constants c_0 and c now follows from the considerations at the beginning of the proof. \square

The corresponding results for the other examples given in Example 5.2 are left as exercises to the reader, who should note however that the action of $\text{O}(\text{Bil}_\Lambda^+(L), q)$ on $\text{Bil}_\Lambda^+(L)$ need not be absolutely irreducible any more.

The next topic is to set the concepts of this chapter into relation with modular lattices as introduced by Quebbemann in [Que95]; cf. also [SSch98] and [Ple98] for surveys.

DEFINITION 5.6.

(i) $\phi \in \text{Bil}_\Lambda^+(L)$ is said to be *k-modular*, for $k \in \mathbf{Z}$, if $(L^\sharp, k\phi)$ is isometric to (L, ϕ) , where $L^\sharp = \{l \in \mathcal{V} \mid \phi(l, L) \subseteq \mathbf{Z}\}$. (Note the Gram matrix of ϕ on L^\sharp is inverse to the Gram matrix on L if one chooses the bases dual to each other.)

(ii) $\text{Bil}_\Lambda^+(L)$ is called *modular* if $\text{Bil}_\Lambda^+(L)$ is special by the maps $\iota: \text{Bil}_\Lambda^+(L) \rightarrow \text{Bil}_\Lambda(L^*)$ and $q: \text{Bil}_\Lambda^+(L) \rightarrow \mathbf{Z}$, cf. Definition 5.1, such that ι is (the restriction to $\text{Bil}_\Lambda^+(L)$ of) an induced equivalence; cf. Definition 4.3.

Clearly, if $\text{Bil}_\Lambda^+(L)$ is modular, each nondegenerate $\phi \in \text{Bil}_\Lambda^+(L)$ is $c_0 q(\phi)$ -modular with c_0 as in Theorem 5.5, and the isometries are all given by the same map. Some examples of two-dimensional modular lattices of covariant forms have already been investigated in the literature, cf. e. g. [Neb98b] where even the Hermite function was discussed for some examples or [Neb96a], where the extremal 3-modular lattice in dimension 24 was discovered. Here the main issue concerns the cases with $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$ or $\mathbf{C}^{2 \times 2}$, since $\mathbf{H}^{2 \times 2}$ cannot occur. Example 6.6 (i) provides an example where $\text{Bil}_\Lambda^+(L)$ is special without being modular. It should be emphasized that induced equivalence between $\text{Bil}_\Lambda(L)$ and $\text{Bil}_\Lambda(L^*)$ is not an uncommon phenomenon. For instance it occurs whenever L and L^* are Λ -isomorphic. That the induced equivalence is ι , is rather rare.

PROPOSITION 5.7. *Let $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2 \times 2}$ and assume $\text{Bil}_\Lambda^-(L) = \mathbf{Z}\psi_1$ and $\text{Bil}_\Lambda^-(L^*) = \mathbf{Z}\psi_2$ with $\psi_1\psi_2 = e \cdot \text{id}_L$ for some natural number e .*

- (i) *If $e = 1$ then $\text{Bil}_\Lambda(L)$ is modular, with ι induced by ψ_2 .*
- (ii) *If ψ_1 and ψ_2 do not have the same elementary divisors, then $\text{Bil}_\Lambda^+(L)$ is not modular.*
- (iii) *If $e^{\dim(L)} \neq \det(\psi_2)^2$ then $\text{Bil}_\Lambda^+(L)$ is not modular.*

Proof. (i) This follows along the lines of Proposition 5.4. That $\text{Bil}_\Lambda(L)$ is mapped onto $\text{Bil}_\Lambda(L^*)$ follows from the fact that $\det(\psi_2) = \pm 1$.

(ii) This is because induced equivalence respects elementary divisors.

(iii) This can be derived from (ii) by taking determinants. It can also be obtained from the observation that ψ_2 induces $e \cdot \iota$. \square

EXAMPLE 5.8.

(i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the e of Proposition 5.7 is 1, 2, 3 and 6. In all cases $\text{Bil}_\Lambda^+(L)$ is modular and c_0 is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.

(ii) In Example 2.2 (ii), choose f_0 to be m -modular for some natural number m . Then $\text{Bil}_\Lambda^+(L \oplus L)$ (in the notation of Example 2.2 (ii)) is modular, where the e of Proposition 5.7 is equal to m , as is c_0 .

To test whether $\text{Bil}_\Lambda^+(L)$ is modular, one can simply compute the images of a \mathbf{Z} -basis of $\text{Bil}_\Lambda^+(L)$ under ι as described in Theorem 5.5 and find a simultaneous isometry of L to L^* (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PLS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with $\mathbf{R} \otimes_{\mathbf{Q}} \text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2 \times 2}$. One such example, involving the Leech lattice with $\text{End}_{\mathcal{A}}(\mathcal{V})$ a non-split quaternion algebra over $\mathbf{Q}[\sqrt{-7}]$, is sketched in the last chapter of [Ple96].

6. SOME THREE-DIMENSIONAL LATTICES OF COVARIANT FORMS

This chapter is devoted to some examples in the case where $\text{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2 \times 2}$ and where the depth of $\text{Bil}_\Lambda(L)$ is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of $\text{Bil}_\Lambda^+(L)$ compare to the orthogonal group of $(\text{Bil}_\Lambda^+(L), q)$? The simplest case is $\text{End}_\Lambda(L) \cong \mathbf{Z}^{2 \times 2}$, where all these questions can be answered.

THEOREM 6.1. *Let $\text{End}_\Lambda(L) \cong \mathbf{Z}^{2 \times 2}$. Then $L = L_0 \oplus L_0$ for some irreducible Λ -lattice L_0 . Let ϕ_0 be the positive definite generator of $\text{Bil}_\Lambda^+(L_0)$. Then c , c_0 , and q , introduced in Theorem 5.5, are as follows.*

- (i) *With respect to a suitable basis of $\text{Bil}_\Lambda^+(L)$, the quadratic form q of Theorem 5.5 becomes $xy - z^2$.*
- (ii) $c = \det(\phi_0)^2$.
- (iii) c_0 *is the exponent of L_0^\sharp/L_0 , i. e. the biggest elementary divisor of a Gram matrix of ϕ_0 .*