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The assumption that the lattice is cocompact is important in the proof but the theorem probably generalizes to non-cocompact lattices. Note also that for many lattices in semi-simple Lie groups, it turns out that the usual cohomology group  $H^2(\Gamma, \mathbf{R})$  vanishes. This is the case for instance for cocompact torsion free lattices in  $SL(n, \mathbf{R})$  for  $n \geq 4$  but more generally for cocompact torsion free lattices in the group of isometries of an irreducible symmetric space of non compact type of rank at least 3 which is not hermitian symmetric (see [7]). In these cases, Theorem 7.9 means that  $H_b^2(\Gamma, \mathbf{R})$  vanishes. Hence, using 6.6, we deduce that every action  $\Gamma$  on the circle has a finite orbit. In other words, Theorems 7.4 and 7.9 are closely related and, indeed they have been proved simultaneously (and independently). It would be very useful to compare the two proofs.

As we have already noticed, the vanishing of the second bounded cohomology group is closely related to the notion of commutator length. If  $\Gamma$  is any group and  $\gamma$  is in the first commutator subgroup  $\Gamma'$ , we denote by  $|\gamma|$  the least integer  $k$  such that  $\gamma$  can be written as a product of  $k$  commutators. We “stabilize” this number and define  $\|\gamma\|$  as  $\lim_{n \rightarrow \infty} |\gamma^n|/n$  (which always exists by sub-additivity). It turns out that for a finitely generated group  $\Gamma$  it is equivalent to say that the second bounded cohomology group  $H_b^2(\Gamma, \mathbf{R})$  injects in the usual cohomology group  $H^2(\Gamma, \mathbf{R})$ , and to say that this “stable commutator norm”  $\|\cdot\|$  vanishes identically [5]. Theorem 7.9 therefore implies that for cocompact higher rank lattices, this stable norm vanishes. The following question is natural:

**PROBLEM 7.10.** *Let  $\Gamma$  be an irreducible lattice as in Theorem 7.4. Does there exist an integer  $k \geq 1$  such that every element of the first commutator subgroup of  $\Gamma$  is a product of  $k$  commutators?*

Recall that by a theorem of Kazhdan, there is no non trivial homomorphism from  $\Gamma$  to  $\mathbf{R}$ ; this is equivalent to the fact that the first commutator group of  $\Gamma$  has finite index in  $\Gamma$ . A positive answer to the previous question would be a strengthening of this fact.

### 7.3 LATTICES IN LINEAR GROUPS

In this section, we prove Theorem 7.4 for lattices in  $SL(n, \mathbf{R})$  ( $n \geq 3$ ). The general case of a semi-simple Lie group is much harder but the proof that we present here contains the main ideas. As a matter of fact, we shall

first concentrate on the case of a lattice  $\Gamma$  in  $SL(3, \mathbf{R})$  and we shall easily deduce the general case of  $SL(n, \mathbf{R})$  later.

Let us first informally describe the structure of the proof. Let  $\Gamma$  be a lattice in  $SL(3, \mathbf{R})$  and consider a homomorphism  $\phi: \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ .

FIRST STEP. In order to prove the theorem, it is enough to show that there is a probability measure  $\mu$  on the circle which is invariant under the group  $\phi(\Gamma)$ .

SECOND STEP (CLASSICAL). A *flag* in  $\mathbf{R}^3$  is a pair consisting of a 2-dimensional (vector) subspace  $E_2$  in  $\mathbf{R}^3$  and a 1-dimensional (vector) subspace  $E_1$  contained in  $E_2$ . Those flags, equipped with the natural topology, define a compact manifold  $Fl$  which is a homogeneous space under the action of  $SL(3, \mathbf{R})$ . Note that in particular,  $\Gamma$  acts on  $Fl$ .

Let  $Prob(\mathbf{S}^1)$  be the space of all probability measures on the circle. Equipped with the weak topology, this is a compact metrizable space on which the group  $\text{Homeo}_+(\mathbf{S}^1)$  acts naturally. The lattice  $\Gamma$  also acts on  $Prob(\mathbf{S}^1)$  via the homomorphism  $\phi$ .

Equip  $Fl$  with the  $\sigma$ -algebra of Lebesgue measurable sets and  $Prob(\mathbf{S}^1)$  with the  $\sigma$ -algebra of Borel sets. In the second step, we construct a measurable map  $\Psi: Fl \rightarrow Prob(\mathbf{S}^1)$  which is equivariant with respect to the actions of  $\Gamma$  on  $Fl$  and  $Prob(\mathbf{S}^1)$ .

In order to prove the theorem, it is enough to show that this map  $\Psi$  takes the same value  $\mu$  almost everywhere with respect to the Lebesgue measure on  $Fl$ . Indeed, by equivariance, this measure  $\mu$  will be invariant by the group  $\phi(\Gamma)$ .

*By way of contradiction, we now assume that  $\Psi$  is not constant on a set of full Lebesgue measure.*

THIRD STEP. Using ergodic properties of the action of  $\Gamma$  on  $Fl$ , we show that there is an integer  $k$  and a measurable map  $\Psi$  as above such that the image of almost every flag in  $Fl$  is the sum of  $k$  Dirac masses on the circle (each with weight  $1/k$ ). Let us denote by  $\mathbf{S}_k^1$  the set of subsets of  $\mathbf{S}^1$  with  $k$  elements so that we can now consider  $\Psi$  as a map from  $Fl$  to  $\mathbf{S}_k^1$ .

FOURTH STEP. Let  $X$  be the space consisting of triples  $(E_2^1, E_2^2, E_2^3)$  of distinct planes in  $\mathbf{R}^3$  intersecting on the same line  $E_1$ . This is again a homogeneous space under the action of  $SL(3, \mathbf{R})$ . An element of  $X$  determines

three flags. Therefore the map  $\Psi$  enables us to define a measurable map  $\Psi^{(3)}: X \rightarrow (\mathbf{S}_k^1)^3$ . We will get a contradiction between the ergodicity of the action of  $\Gamma$  on  $X$  and the non ergodicity of the action of  $\Gamma$  on the set of triples of points: a triple of points on  $\mathbf{S}^1$  can be positively or negatively ordered on the circle and this is invariant under  $\text{Homeo}_+(\mathbf{S}^1)$ .

We now give the detailed proof.

**FIRST STEP: FINDING AN INVARIANT MEASURE.** Suppose that there is a probability measure  $\mu$  on the circle which is invariant under  $\phi(\Gamma)$ .

We know that the rotation number mapping  $\rho: \text{Homeo}_+(\mathbf{S}^1) \rightarrow \mathbf{R}/\mathbf{Z}$  is not a homomorphism. However by 6.18, the restriction to the subgroup consisting of homeomorphisms preserving a given measure  $\mu$  is a homomorphism. It follows that the map  $\gamma \in \Gamma \mapsto \rho(\phi(\gamma)) \in \mathbf{R}/\mathbf{Z}$  is a homomorphism. According to the result of Kazhdan that we mentioned several times already,  $\Gamma$  is finitely generated and every homomorphism from  $\Gamma$  to  $\mathbf{R}$  is trivial. It follows that the image of the restriction of  $\rho$  to  $\Gamma$  is a finite cyclic subgroup  $\mathbf{Z}/k\mathbf{Z}$ . Consider the kernel  $\Gamma_0$  of this homomorphism: this is a subgroup of index  $k$  of  $\Gamma$ , hence a lattice in  $\text{SL}(3, \mathbf{R})$ . We claim that the support of  $\mu$  is fixed pointwise by  $\Gamma_0$ . This follows from the fact that for every homeomorphism of the circle with zero rotation number, the support of every invariant measure is contained in the set of fixed points. Hence every point in the support of  $\mu$  has a finite orbit under  $\phi(\Gamma)$ . This is the conclusion of Theorem 7.4.

**SECOND STEP: FURSTENBERG MAP.** This step is classical in the study of actions of lattices and is due to Furstenberg [23].

**PROPOSITION 7.11.** *There is a Lebesgue measurable map  $\Psi: Fl \rightarrow \text{Prob}(\mathbf{S}^1)$  which is equivariant under the actions of  $\Gamma$  on  $Fl$  and  $\text{Prob}(\mathbf{S}^1)$ .*

*Proof.* We observed that  $Fl$  is homogeneous under the action of  $\text{SL}(3, \mathbf{R})$ . The stabilizer of the flag consisting of the line spanned by  $(1, 0, 0)$  and the plane generated by  $(1, 0, 0)$  and  $(0, 1, 0)$  is the group  $B$  of upper triangular matrices. Therefore we can identify  $Fl$  and the homogeneous space  $\text{SL}(3, \mathbf{R})/B$ .

Note that the group  $B$  is solvable. Hence  $B$  is amenable and there is a linear form  $m$  on  $L^\infty(B, \mathbf{R})$  which is non negative on non negative functions, takes the value 1 on the constant function 1 and is invariant under left translations. It turns out that it is possible to choose  $m$  in such a way that it is a measurable

function (see [55]). In other words, if  $f_\lambda \in L^\infty(B, \mathbf{R})$  depends measurably on a parameter  $\lambda$  in  $[0, 1]$ , the function  $\lambda \mapsto m(f_\lambda)$  is Lebesgue measurable.

Coming back to our problem, we first observe that there are measurable maps  $\Psi_0: \mathrm{SL}(3, \mathbf{R}) \rightarrow \mathrm{Prob}(\mathbf{S}^1)$  which are  $\Gamma$ -equivariant. This follows from the fact that the action of  $\Gamma$  on  $\mathrm{SL}(3, \mathbf{R})$  by left translations has a fundamental domain; we define  $\Psi_0$  in an arbitrary measurable way on this fundamental domain and we can therefore define it everywhere using the equivariance.

To complete the proof of the proposition, we modify  $\Psi_0$  to make it invariant under right translations under  $B$ . Of course, we use the mean  $m$ . We define  $\Psi: \mathrm{SL}(3, \mathbf{R}) \rightarrow \mathrm{Prob}(\mathbf{S}^1)$  in the following way. If  $g \in \mathrm{SL}(3, \mathbf{R})$ , the probability  $\Psi(g)$  is defined by its value on a continuous function  $u: \mathbf{S}^1 \rightarrow \mathbf{R}$ :

$$\int_{\mathbf{S}^1} u d\Psi(g) = m(x \in B \mapsto \int_{\mathbf{S}^1} u d\Psi_0(gx)).$$

By construction,  $\Psi$  is measurable and invariant by right translations by  $B$ ; this defines another measurable map  $\Psi: Fl \simeq \mathrm{SL}(3, \mathbf{R})/B \rightarrow \mathrm{Prob}(\mathbf{S}^1)$  which is  $\Gamma$ -equivariant, as required.  $\square$

THIRD STEP: THE MAP  $\Psi$  TO DIRAC MASSES. As mentioned above, we now assume by contradiction that the map  $\Psi$  is not constant on a subset of full Lebesgue measure.

PROPOSITION 7.12. *There exist an integer  $k \geq 1$  and a map  $\Psi: Fl \rightarrow \mathbf{S}_k^1$  to the set of subsets of  $\mathbf{S}^1$  with  $k$  elements which is Lebesgue measurable and  $\Gamma$ -invariant.*

In order to prove the proposition, we first recall an important ergodic theorem due to Moore that we shall use repeatedly (see for instance [72]). Let  $Y = G/H$  be a homogeneous space of a semi-simple Lie group  $G$ . Assume that  $G$  is connected, has a finite center and has no compact factor. Assume moreover that  $H$  is non compact. Let  $\Gamma$  be an irreducible lattice in  $G$ . Then the action of  $\Gamma$  on  $Y$  is ergodic with respect to the Lebesgue measure (class), i.e. every measurable function on  $Y$  which is  $\Gamma$ -invariant is constant almost everywhere.

For instance, the stabilizer  $B$  of a flag is non compact. *The action of  $\Gamma$  on  $Fl$  is ergodic.*

As another example, let us consider the space  $Y$  of pairs of flags of  $\mathbf{R}^3$  which are in general position. For such a pair of flags, there are three non coplanar lines  $E_1^1, E_1^2, E_1^3$  such that the first flag is given by the line  $E_1^1$  and

the plane spanned by  $E_1^1$  and  $E_1^2$  and the second flag is given by the line  $E_1^3$  and the plane spanned by  $E_1^2$  and  $E_1^3$ . Since  $SL(3, \mathbf{R})$  acts transitively on the space of triples of non coplanar lines, it follows that  $Y$  is a homogeneous space of  $SL(3, \mathbf{R})$ . The stabilizer of an element of  $Y$  is the stabilizer of a triple of non coplanar lines: it is clearly non compact. Consequently, the action of  $\Gamma$  on  $Y$  is ergodic. Since the set of pairs of flags in general position has full Lebesgue measure in the set of pairs of flags, we deduce that  $\Gamma$  acts ergodically on the set of pairs of flags of  $\mathbf{R}^3$ .

However, the reader will easily check that this cannot be generalized to the set of triples of flags: the action of  $SL(3, \mathbf{R})$  is not transitive on the set of triples of flags in general position.

In order to prove Proposition 7.12, we analyze the action of  $\Gamma$  on the space of pairs of probability measures on the circle.

If  $\mu$  is a probability on the circle, we define  $atom(\mu)$  as the sum of the masses of the atoms of  $\mu$  (i.e. those points  $x$  such that  $\mu(\{x\}) > 0$ ). This is a measurable function on  $Prob(\mathbf{S}^1)$  which is invariant under the action of  $Homeo_+(\mathbf{S}^1)$ . The map:

$$d \in Fl \mapsto atom(\Psi(d)) \in [0, 1]$$

is a measurable  $\Gamma$ -invariant function. Using the ergodicity result that we mentioned above, this function is constant almost everywhere.

*Assume first that this constant is not zero.* This means that the image of almost every flag under  $\Psi$  has at least one atom.

Let  $\alpha > 0$  be a positive real number. For each probability measure  $\mu$  on the circle, consider the points  $x$  such that  $\mu(\{x\}) > \alpha$ . Of course, the number of those points  $x$  is finite (possibly zero). Denote this number by  $N(\mu, \alpha)$ . The map  $d \in Fl \mapsto N(\Psi(d), \alpha) \in \mathbf{N}$  is measurable and  $\Gamma$ -invariant; it is therefore constant, equal to some integer  $N_\alpha$  almost everywhere. Since we assume that for almost every  $d$  the probability  $\Psi(d)$  has at least one atom, we can choose some  $\alpha$  in such a way that  $N_\alpha$  is an integer  $k \geq 1$ . This enables us to construct a map (defined almost everywhere) from  $Fl$  to the set of subsets of  $\mathbf{S}^1$  with  $k$  elements, sending the flag  $d$  to the  $k$  atoms of  $\Psi(d)$  having a mass greater than or equal to  $\alpha$ . Changing our notation, we shall call this new map  $\Psi$ : it satisfies Proposition 7.12 which is therefore proved, if almost every  $\Psi(d)$  has at least one atom.

*We now assume that for almost every  $d$ , the probability  $\Psi(d)$  has no atom.*

We shall show that under this assumption, almost all the measures  $\Psi(d)$  have the same support.

Let  $\mu_1$  and  $\mu_2$  be two probability measures on the circle with no atom. Define  $D(\mu_1, \mu_2)$  as the maximum of the  $\mu_2$ -measures of the connected components of the complement of the support of  $\mu_1$ . If  $D(\mu_1, \mu_2) = 0$ , the support of  $\mu_1$  contains the support of  $\mu_2$ . The map

$$(d_1, d_2) \in Fl^2 \mapsto D(\Psi(d_1), \Psi(d_2)) \in [0, 1]$$

is defined almost everywhere and is  $\Gamma$ -invariant. Using the same ergodicity result as before, we deduce that it is constant almost everywhere. *We claim that this constant  $\delta$  is 0.*

Suppose on the contrary that  $\delta > 0$ . Using Fubini's theorem, we can find a measurable part  $\Omega \subset Fl$  such that:

- $\Omega$  has full Lebesgue measure.
- If  $d \in \Omega$ , the probability  $\Psi(d)$  has no atom.
- If  $d \in \Omega$ , then  $D(d, d') = \delta$  for almost every  $d'$  in  $Fl$ .
- If  $d \in \Omega$ , then  $\Psi(d)$  belongs to the support of the measure  $\Psi_*(\text{Lebesgue})$  on the compact metrizable space  $Prob(\mathbf{S}^1)$ .

Fix a point  $d \in \Omega$ . We can find a sequence  $d_i \in \Omega$  such that  $\Psi(d_i) = \mu_i$  converges towards  $\Psi(d) = \mu$ . The probability measures  $\mu_i$  have no atoms and  $D(\mu_i, \mu) = \delta$ . This means that there is a component  $I_i$  of the complement of the support  $\text{supp}(\mu)$  such that  $\mu_i(I_i) = \delta$ . If the sequence of lengths of  $I_i$  converges to 0, we can assume that the sequence of intervals  $I_i$  shrinks to a point  $p$ . This implies that the point  $p$  is an atom of  $\mu$ , contradicting our assumption. Therefore we can assume (after taking a subsequence) that the intervals  $I_i$  all coincide with some interval  $I$ . Since we know that the endpoints of  $I$  are not atoms of  $\mu$ , that the sequence  $\mu_i$  converges weakly to  $\mu$ , and that  $\mu_i(I) = \delta$ , it follows that  $\mu(I) = \delta$ . This contradicts the fact that  $I$  is in the complement of the support of  $\mu$ .

We showed that  $\delta = 0$ . This means that for almost every pair of flags  $(d, d')$ , we have  $D(\Psi(d), \Psi(d')) = 0$ . Therefore, for almost every pair of flags  $(d, d')$ , the probability measures  $\Psi(d)$  and  $\Psi(d')$  have the same support. In other words, *there exists a compact set  $K \subset \mathbf{S}^1$  with no isolated point, such that for almost every flag  $d$ , the support of  $\Psi(d)$  is equal to  $K$ .*

Each connected component of  $\mathbf{S}^1 - K$  is an open interval. Collapsing the closure of these intervals to a point, we get a space homeomorphic to a circle. Therefore, there exists a continuous  $\pi: \mathbf{S}^1 \rightarrow \mathbf{S}^1$  such that each fiber of  $\pi$  is a point or the closure of a component of the complement of  $K$ . If  $\mu$  is

a measure with no atom whose support is  $K$ , the direct image  $\pi_*(\mu)$  is a measure on the circle with no atom and full support on the circle.

Using  $\pi_*$ , we get a map  $\bar{\Psi}$  from  $Fl$  to the space of probability measures on the circle with no atom and full support which is  $\Gamma$ -equivariant with respect to the minimal action  $\bar{\phi}$  associated to  $\phi$  (see 5.8).

The space of probability measures with no atoms and full support on the circle is a homogeneous space under the action of  $\text{Homeo}_+(\mathbf{S}^1)$  and the stabilizer of the Lebesgue measure is of course  $\text{SO}(2)$ . This space can therefore be identified with the quotient  $\text{Homeo}_+(\mathbf{S}^1)/\text{SO}(2)$ . The group  $\text{Homeo}_+(\mathbf{S}^1)$ , as any metrizable topological group, can be equipped with a left invariant metric, that we can average under the action of  $\text{SO}(2)$  to produce a left invariant metric  $dist$  on  $\text{Homeo}_+(\mathbf{S}^1)/\text{SO}(2)$ . In practice, we could simply define  $dist(\mu_1, \mu_2)$  as the supremum of  $|\mu_1(I) - \mu_2(I)|$  where  $I$  runs through the collection of intervals on the circle: it is easy to check that this metric indeed defines the weak topology when restricted to the set of probability measures with no atom and full support.

For almost every pair of flags  $(d, d')$  the distance  $dist(\bar{\Psi}(d), \bar{\Psi}(d'))$  defines a  $\Gamma$ -invariant function of pairs of flags; it is therefore constant almost everywhere. Using the same argument as above, we see that this constant is 0, which means that the map  $\bar{\Psi}$  is constant almost everywhere. Of course, two probability measures with no atom and with support in  $K$  which have the same image under  $\pi_*$  have to coincide so that we deduce that  $\Psi$  is constant almost everywhere. We have found a probability measure on the circle which is invariant under  $\phi(\Gamma)$ . This is a contradiction with our initial assumption and proves 7.12.

**FOURTH STEP: CYCLIC ORDERING ON TRIPLES OF POINTS ON A CIRCLE.** In order to explain the general idea, we assume first that the integer  $k$  that we introduced is equal to 1. In other words, we have a  $\Gamma$ -invariant map  $\Psi: Fl \rightarrow \mathbf{S}^1$  defined almost everywhere which is not constant on a set of full Lebesgue measure.

As explained above, let  $X$  denote the space of triples  $(E_2^1, E_2^2, E_2^3)$  of distinct planes in  $\mathbf{R}^3$  intersecting on the same line  $E_1$ . This is again a homogeneous space under  $\text{SL}(3, \mathbf{R})$  and the stabilizer of a point in  $X$  is clearly non compact. We deduce from Moore ergodicity theorem that the action of  $\Gamma$  on  $X$  is ergodic. Since a point of  $X$  determines three flags, we can define a measurable  $\Gamma$ -equivariant map  $\Psi^{(3)}: X \rightarrow (\mathbf{S}^1)^3$  (defined almost everywhere). Indeed, let us consider the projection  $pr: Fl \rightarrow \mathbf{RP}^2$  from  $Fl$  to the real projective plane mapping a flag  $E_1 \subset E_2$  to the line  $E_1 \subset \mathbf{R}^3$ . The space  $X$

is therefore the space of triples of flags having the same projection under  $pr$ . It follows from Fubini's theorem that for every subset of full measure in  $Fl$ , the set of triples of elements of this set having the same projection under  $pr$  has full measure in  $X$ : this is exactly what we need to define  $\Psi^{(3)}$ .

The space  $(\mathbf{S}^1)^3$  can be decomposed into disjoint parts, invariant under the action of  $\text{Homeo}_+(\mathbf{S}^1)$ :

i) Triples of the form  $(x, x, x)$ .

ii) Triples consisting of two distinct points. In turn, this set can be decomposed into three parts: the spaces of triples of the form  $(x, x, z)$ , resp.  $(x, y, x)$ , resp.  $(x, y, y)$ .

iii) Triples  $(x, y, z)$  of distinct elements on the circle whose cyclic ordering is positive, *i.e.* such that the interval positively oriented from  $x$  to  $y$  does not contain  $z$ .

iv) Triples  $(x, y, z)$  of distinct elements on the circle whose cyclic ordering is negative.

Inverse images of these six parts under  $\Psi^{(3)}$  are measurable and disjoint  $\Gamma$ -invariant sets and therefore have to be either of measure 0 or of full Lebesgue measure. This means that there is a subset  $\Omega \subset X$  of full measure whose image is contained in one of the six parts that we described. We claim that this is not possible.

Observe that the symmetric group  $\mathfrak{S}_3$  of permutations of three objects acts on  $X$  and on  $(\mathbf{S}^1)^3$ , permuting respectively flags and points. Note that these actions commute with the actions of  $\Gamma$  on  $X$  and  $(\mathbf{S}^1)^3$ . Of course  $\Psi^{(3)}$  is equivariant with respect to these action of  $\mathfrak{S}_3$ .

It follows that the part which contains  $\Psi^{(3)}(\Omega)$  has to be invariant under  $\mathfrak{S}_3$ . Among the 6 parts that we described, only the first one has this property. This means that the map  $\Psi: Fl \rightarrow \mathbf{S}^1$  factors through the projection  $pr: Fl \rightarrow \mathbf{RP}^2$ . In other words, almost everywhere, the image of a flag by  $\Psi$  depends only on the line associated to the flag and not on its plane.

Exactly in the same way, we could have defined a space  $X'$  consisting of triples of flags having the same plane, *i.e.* having the same projection in the dual projective plane. The same proof shows that almost everywhere  $\Psi$  depends only on the plane of a flag and not on its line.

This implies that  $\Psi$  is constant almost everywhere and gives the contradiction we were looking for when  $k = 1$ .

*When  $k > 1$ , we shall use a similar idea.*

Recall that we denote by  $\mathbf{S}_k^1$  the space of subsets  $A$  of the circle with  $k$  elements. Given two elements  $(A_1, A_2, A_3)$  and  $(A'_1, A'_2, A'_3)$  of  $(\mathbf{S}_k^1)^3$ , we say

that they have the same cyclic ordering if there is an orientation preserving homeomorphism  $h$  of the circle such that  $h(A_1) = A'_1, h(A_2) = A'_2, h(A_3) = A'_3$ . This gives a partition of  $(\mathbf{S}_k^1)^3$  into finitely many parts invariant under the action of  $\text{Homeo}_+(\mathbf{S}^1)$ . As before, it follows that there is a subset  $\Omega$  of full measure in  $X$  such that  $\Psi(\Omega)$  is contained in one of these subsets. Using again the action of  $\mathfrak{S}_3$  we conclude that this subset consists of triples  $(A_1, A_2, A_3)$  which have the same cyclic ordering as  $(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)})$  for every element  $\sigma \in \Sigma_3$ . Therefore, for every  $\sigma$ , there is an orientation preserving homeomorphism  $h_\sigma$  such that  $h_\sigma(A_i) = A_{\sigma(i)}$  for  $i = 1, 2, 3$ . Let  $A$  be the union of  $A_1, A_2$  et  $A_3$ : this is a set with  $N \leq 3k$  elements. Orientation preserving homeomorphisms globally preserving  $A$  must induce a cyclic permutation of its elements. In particular, the commutator of two elements  $h_\sigma$  must fix each element of  $A$  since cyclic permutations commute. As the cyclic permutation  $\sigma = (1, 2, 3)$  is a commutator in  $\mathfrak{S}_3$ , the homeomorphism  $h_{(1,2,3)}$  acts trivially on  $A$ . Since we know that  $h_{(1,2,3)}(A_1) = A_2, h_{(1,2,3)}(A_2) = A_3$  and  $h_{(1,2,3)}(A_3) = A_1$ , we have  $A_1 = A_2 = A_3$ . We showed that there exists a measurable subset of full measure  $\Omega \subset X$  such that the image  $\Psi(\Omega)$  consists of triples of the form  $(A, A, A)$ . Exactly as we did in the case  $k = 1$ , we conclude that  $\Psi$  is constant almost everywhere and this is a contradiction.

This is the end of the proof of Theorem 7.4 for lattices in  $\text{SL}(3, \mathbf{R})$ .

Remark that the core of the proof is the incompatibility between two facts. The group  $\text{Homeo}_+(\mathbf{S}^1)$  does not act transitively on generic triples of points on the circle but  $\text{SL}(3, \mathbf{R})$  does act transitively on  $X$ . Note that the existence of an element of  $\text{SL}(3, \mathbf{R})$  fixing a line and permuting arbitrarily three planes containing this line, means that the real projective plane is not orientable.

*The proof for a lattice  $\Gamma$  in  $\text{SL}(n, \mathbf{R})$  ( $n \geq 3$ ) is very similar. For every sequence of integers,  $1 \leq i_1 < i_2 < \dots < i_l \leq n$ , we consider the space  $Fl_{i_1, \dots, i_l}$  of flags of type  $(i_1, \dots, i_l)$ , i.e. sequences of vector sub-spaces  $E_{i_1} \subset E_{i_2} \subset \dots \subset E_{i_l} \subset \mathbf{R}^n$  with  $\dim E_{i_j} = i_j$  ( $j = 1, \dots, l$ ). This is a homogeneous space under the action of  $\text{SL}(n, \mathbf{R})$ . The space of complete flags, i.e.  $Fl = Fl_{1,2, \dots, n}$  is equipped with projections  $pr_j$  on incomplete flag spaces  $Fl_{1,2, \dots, \hat{j}, \dots, n}$  where the index  $j$  does not appear. The space  $X_j$  consisting of distinct triples of flags  $Fl$  having the same projection under  $pr_j$  is again a homogeneous space of  $\text{SL}(n, \mathbf{R})$ , with non compact stabilizer.*

Now, the proof is the same as before. We first construct an equivariant map  $\Psi$  from  $Fl$  to  $\text{Prob}(\mathbf{S}^1)$  (same proof). Assuming by contradiction that  $\Psi$  is not constant almost everywhere, we get another map, still denoted by  $\Psi$  from  $Fl$  to  $\mathbf{S}_k^1$  (same proof). For each  $j = 1, \dots, n$ , we consider the

corresponding map  $\Psi_j^{(3)}: X_j \rightarrow \mathbf{S}_k^1$  and we show, as above, that the image of this map consists almost everywhere of triples of the form  $(A, A, A)$ . It follows that for each  $j = 1, \dots, n$  and on a subset of full measure, the image of a flag by  $\Psi$  depends only on its projection by  $pr_j$ . Since this is true for every  $j$ , this means that  $\Psi$  is constant almost everywhere. This is a contradiction and finishes the proof of Theorem 7.4 for lattices in  $SL(n, \mathbf{R})$ .

Of course, these proofs immediately generalize to lattices in complex or quaternionic special linear groups  $SL(n, \mathbf{C})$  and  $SL(3, \mathbf{H})$  (for  $n \geq 3$ ).

#### 7.4 SOME GROUPS THAT DO ACT...

We saw that many higher rank lattices don't act on the circle. To conclude these notes, we give some more examples of "big" groups acting on the circle. Let  $\Sigma$  be a compact oriented surface of genus  $g \geq 2$  and  $x \in \Sigma$  be some base point. The fundamental group  $\pi_1(\Sigma, x)$  is a classical example of a hyperbolic group in the sense of Gromov (see for instance [27]). The boundary of this group is a topological circle: indeed  $\pi_1(\Sigma, x)$  acts freely and cocompactly on the Poincaré disc so that  $\pi_1(\Sigma, x)$  is quasi-isometric to the Poincaré disc. Consequently, the automorphism group  $\text{Aut}(\pi_1(\Sigma, x))$  acts naturally on the circle. This action is very interesting and has been very much studied. See for instance [21]. Note that  $\text{Aut}(\pi_1(\Sigma, x))$  contains the group of inner conjugacies and that the quotient  $\text{Out}(\pi_1(\Sigma, x))$  is the *mapping class group* of the surface (*i.e.* the group of isotopy classes of homeomorphisms of the surface):

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \text{Aut}(\pi_1(\Sigma, x)) \longrightarrow \text{Out}(\pi_1(\Sigma, x)) \longrightarrow 1.$$

Fix an element  $f$  of infinite order in this mapping class group and consider the group  $\Gamma_f$  which is the inverse image of the group generated by  $f$  in the previous exact sequence. We have an exact sequence:

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \Gamma_f \longrightarrow \mathbf{Z} \longrightarrow 1.$$

This group  $\Gamma_f$  is the fundamental group of the 3-manifold which fibers over the circle and whose monodromy is given by the class  $f$ . Thurston showed that if  $f$  is of pseudo-Anosov type, then this 3-manifold is hyperbolic. In particular, for such a choice of  $f$ , the group  $\Gamma_f$  embeds as a discrete cocompact subgroup of the isometry group of the hyperbolic 3-ball, isomorphic to  $\text{PSL}(2, \mathbf{C})$ . This construction provides many examples of faithful actions of (rank 1) lattices on the circle. In [68] Thurston constructs faithful actions of the fundamental group of many hyperbolic 3-manifolds on the circle.

Suppose now that  $\Sigma$  has one boundary component  $\partial\Sigma$ . Choose the base point on the boundary and equip  $\Sigma$  with a metric with curvature  $-1$  and