7. HIGHER RANK LATTICES

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homomorphism from Γ to $\mathbf{R}$. Of course, for any γ in Γ, the projection of $\psi(\gamma)$ in $\mathbf{R}/\mathbf{Z}$ is nothing more than the rotation number of $\phi(\gamma)$. Summing up, *with these algebraic conditions on the group* Γ, *any action of* Γ *on the circle determines canonically a quasi-homomorphism* $\psi\colon \Gamma \to \mathbf{R}$ *which is a lift of the rotation number map.*

A specific example is the modular group $\mathrm{PSL}(2, \mathbf{Z})$. As a group, it is isomorphic to the free product of two cyclic groups: $\mathrm{PSL}(2, \mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/3\mathbf{Z}$ (see for instance [61]). Of course there is no non-trivial homomorphism from this group to $\mathbf{R}$ since it is generated by two elements of finite order. In the same way, its second real cohomology group is trivial (this follows for instance from the Mayer-Vietoris exact sequence since finite groups have trivial cohomology over the reals). We deduce that every action of $\mathrm{PSL}(2, \mathbf{Z})$ on the circle yields a well defined quasi-homomorphism $\psi\colon \mathrm{PSL}(2, \mathbf{Z}) \to \mathbf{R}$ lifting the rotation number. If we start with the canonical action of $\mathrm{PSL}(2, \mathbf{Z})$ on the circle $\mathbf{RP}^1$, the rotation numbers are not interesting: the only elliptic elements in $\mathrm{PSL}(2, \mathbf{Z})$ have order 2 and 3 so that the rotation number of elements in $\mathrm{PSL}(2, \mathbf{Z})$ are $0, 1/2, 1/3, 2/3 \in \mathbf{R}/\mathbf{Z}$. However the quasi-homomorphism $\Psi\colon \mathrm{PSL}(2, \mathbf{Z}) \to \mathbf{R}$ that we get is very interesting and has been studied in many different contexts: it is called the *Rademacher function.* The explicit formula giving Ψ as a function of the entries of a matrix in $\mathrm{PSL}(2, \mathbf{Z})$ involves the so called Dedekind sums which are important in number theory. We refer to [4] for a description of Ψ and a bibliography on this very nice subject.

7. Higher rank lattices

In this section, we study the problem of determining which lattices in semi-simple groups can act on the circle.

Let G be any Lie group and $\mathfrak{G}$ be its Lie algebra. The *real rank* of G is the maximal dimension of an abelian subalgebra $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ the linear operator $ad(a)\colon \mathfrak{G} \to \mathfrak{G}$ is diagonalizable over $\mathbf{R}$. For instance, the real rank of $\mathrm{SL}(n, \mathbf{R})$ is $n-1$: its Lie algebra consists of traceless matrices and contains the abelian diagonal traceless matrices. A *lattice* in a Lie group G is a discrete subgroup Γ such that the quotient G/Γ has finite measure with respect to a right invariant Haar measure. A lattice in a *semi-simple group* is called *reducible* if we can find two normal subgroups G_1, G_2 in G, connected and non trivial, which generate G, whose intersection is contained in the (discrete) center of G, and such that $(G_1 \cap \Gamma).(G_2 \cap \Gamma)$ has finite index in Γ. Otherwise, we say that Γ is *irreducible*. Note that lattices in simple Lie groups are obviously irreducible.

The first example of a lattice is $\mathrm{SL}(n, \mathbf{Z})$ in $\mathrm{SL}(n, \mathbf{R})$: the corresponding quotient has finite volume (but is not compact).

Another example to keep in mind is the following. Consider the field $\mathbf{Q}(\sqrt{2})$ and its ring of integers $\mathcal{O} = \mathbf{Z}[\sqrt{2}]$. The field $\mathbf{Q}(\sqrt{2})$ has two embeddings in $\mathbf{R}$ given by $a + b\sqrt{2} \in \mathbf{Q}(\sqrt{2}) \mapsto a \pm b\sqrt{2} \in \mathbf{R}$. This gives two embeddings of the group $\mathrm{SL}(2, \mathcal{O})$ in $\mathrm{SL}(2, \mathbf{R})$. The images of these embeddings are dense but the embedding of $\mathrm{SL}(2, \mathcal{O})$ in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ has a discrete image which is an irreducible lattice in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ (whose real rank is 2). Of course, we can construct many more examples using this kind of arithmetic construction: Borel showed for instance that any semi-simple Lie group (with no compact factor) contains at least an irreducible lattice (and even a cocompact one).

Note also that if a compact oriented manifold M of dimension n admits a metric with constant negative curvature, its universal cover is identified with the hyperbolic space H^n of dimension n. It follows that the fundamental group Γ of M is a discrete cocompact subgroup of the group of positive isometries of H^n which is the simple Lie group $\mathrm{SO}_0(n, 1)$. These examples provide lattices in real rank 1 simple Lie groups.

For the theory of lattices in Lie groups, we refer to [48, 72].

7.1 Witte's theorem

In [70], Witte proves the following remarkable theorem:

THEOREM 7.1 (Witte). *Let Γ be a finite index subgroup of* $\mathrm{SL}(n, \mathbf{Z})$ *for* $n \geq 3$. *Then any homomorphism* $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image.*

The proof will be derived from the following

THEOREM 7.2 (Witte). *A finite index subgroup of* $\mathrm{SL}(n, \mathbf{Z})$ *for* $n \geq 3$ *is not left orderable.*

Proof. It suffices to prove it for a finite index subgroup Γ of $\mathrm{SL}(3, \mathbf{Z})$ since a subgroup of a left ordered group is of course left ordered. Suppose by contradiction that there is a left invariant total order $\preceq$ on Γ. Choose some integer $k \geq 1$ so that the following six elementary matrices belong to Γ :

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}.$$

It is easy to check the following relations between these matrices. Taking indices modulo 6, for every i the matrices a_i and a_{i+1} commute and the commutator of a_{i-1} and a_{i+1} is $a_i^{\pm k}$. Fix some i and let us analyze the structure of $\preceq$ on the group H_i generated by a_{i-1}, a_i, a_{i+1}. Allowing ourself to replace a_{i-1} or a_{i+1} by their inverses and to permute them, we can define three elements α, β, γ such that $\{\alpha, \beta\} = \{a_{i-1}^{\pm 1}, a_{i+1}^{\pm 1}\}$ and $\gamma = a_i^{\pm k}$ and such that the following conditions are satisfied:

$$\alpha\gamma = \gamma\alpha \quad ; \quad \beta\gamma = \gamma\beta \quad ; \quad \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^{-1}$$

$$1 \prec \alpha \quad ; \quad 1 \prec \beta \quad ; \quad 1 \prec \gamma$$

(1 denotes the identity element). If ξ is an element of Γ, we set $|\xi| = \xi$ if $1 \preceq \xi$ and ξ^{-1} otherwise. If two elements ξ, ζ in Γ are such that $1 \prec \xi$ and $1 \prec \zeta$, we write $\xi \ll \zeta$ if for every integer $n \geq 1$, we have $\xi^n \prec \zeta$. We claim that $\gamma \ll \alpha$ or $\gamma \ll \beta$ (which implies that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$). Indeed, suppose that there is some integer $n \geq 1$ such that $\alpha \prec \gamma^n$ and $\beta \prec \gamma^n$ and let us compute

$$\delta_m = \alpha^m \beta^m (\alpha^{-1}\gamma^n)^m (\beta^{-1}\gamma^n)^m .$$

Since δ_m is a product of elements in Γ which are bigger than 1, we have $1 \prec \delta_m$. Now the product defining δ_m can easily be estimated since we know that γ commutes with α and β and that interchanging the order of an α and a β is compensated by the introduction of a γ. We find

$$\delta_m = \gamma^{-m^2+2mn} .$$

Since $1 \prec \gamma$, we know that γ to a negative power is less than 1. For m big enough, we get $\delta_m \prec 1$. This is a contradiction.

Coming back to our six matrices a_i, we find that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$. If we assume for instance $|a_1| \ll |a_2|$, we therefore deduce cyclically $|a_1| \ll |a_2| \ll |a_3| \ll |a_4| \ll |a_5| \ll |a_6| \ll |a_1|$, and this is a contradiction. □

Let us now prove Theorem 7.1 using similar ideas. Of course, Theorem 7.2 means that a finite index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$ does not act faithfully on the line (by orientation preserving homeomorphisms).

Consider first a torsion free finite index subgroup Γ of $SL(3, \mathbf{Z})$ and suppose by contradiction that there is an action $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ with infinite image. According to an important theorem, due to Margulis, every normal subgroup of a lattice in a simple Lie group of rank at least 2 is either of finite index or is finite (see [48, 64]). It follows that the action ϕ is faithful.

As in the proof of Theorem 7.2, choose an integer k such that the matrices $(a_i)_{i=1\dots 6}$ are in Γ. Note that the group H_i generated by a_{i-1}, a_i, a_{i+1} is nilpotent, hence amenable, so that the rotation number is a homomorphism when restricted to H_i. Since $a_i^{\pm k}$ is a commutator, it follows that the rotation numbers of all $\phi(a_i)$ vanish. Define A_i as being the unique lift of $\phi(a_i)$ whose translation number is 0. We claim that the elements A_i of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ also satisfy the relations that for every i the homeomorphisms A_i and A_{i+1} commute and the commutator of A_{i-1} and A_{i+1} is $A_i^{\pm k}$. Indeed $A_i A_{i+1} A_i^{-1} A_{i+1}^{-1}$ and $A_{i+1} A_{i-1} A_{i+1}^{-1} A_{i-1}^{-1} A_i^{\mp k}$ project on the identity and have translation number 0 since the inverse image of H_i in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is nilpotent and the restriction of τ to this group is a homomorphism. Consider now the (left ordered) group of homeomorphisms of the line generated by the A_i. We can reproduce exactly the same argument that we used in Theorem 7.2 to get a contradiction.

Consider finally the general case of an action $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ of a finite index subgroup of $SL(n, \mathbf{Z})$ $(n \geq 3)$. Replacing Γ by a finite index subgroup, we can assume that Γ is torsion free. Of course, $SL(3, \mathbf{Z})$ is the subgroup of $SL(n, \mathbf{Z})$ consisting of matrices preserving $\mathbf{Z}^3 \simeq \mathbf{Z}^3 \times \{0\} \subset \mathbf{Z}^n$ and Γ intersects $SL(3, \mathbf{Z})$ on a subgroup of finite index in $SL(3, \mathbf{Z})$. Since we have already dealt with the case $n = 3$, the kernel of ϕ contains a subgroup of finite index in the infinite group $\Gamma \cap SL(3, \mathbf{Z})$. By the theorem of Margulis that we mentioned, the kernel of ϕ is a subgroup of finite index in Γ so that the image of ϕ is a finite group. Theorem 7.1 is proved.

It turns out that the arguments used in this proof can be extended to a family of lattices more general than finite index subgroups of $SL(n, \mathbf{Z})$ for $n \geq 3$. The general situation in which Witte proves his theorem is for *arithmetic* lattices in algebraic semi-simple groups of $\mathbf{Q}$-rank at least 2. We will not define this concept and refer to the original article by Witte. Note however that the method of proof cannot be generalized to an arbitrary lattice since it uses strongly the existence of nilpotent subgroups (which don't exist for example if the lattice is cocompact). However, this strongly suggests the following:

PROBLEM 7.3. *Is it true that no lattice in a simple Lie group of real rank at least 2 is left orderable?*

7.2 Actions of higher rank lattices

We now study actions of the most general higher rank lattices on the circle. Most of this section is an expansion (and a translation) of a small part of [26] to which we refer for more information.

THEOREM 7.4 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any action of Γ on the circle has a finite orbit.*

Of course, in such a situation a subgroup of finite index in Γ acts with a fixed point so that, deleting this fixed point, we get an action of a subgroup of finite index acting on the line. Recall our question 7.3 concerning ordering on lattices; it can be reformulated in the following way:

PROBLEM 7.5. *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Is it true that any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image?*

These notes only deal with actions by homeomorphisms and we decided not to discuss properties connected with smooth diffeomorphisms. However, we mention that the previous question has a positive answer assuming some smoothness.

THEOREM 7.6 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any homomorphism from Γ to the group of C^1-diffeomorphisms of the circle has a finite image.*

This theorem is an immediate consequence of 7.4 and of two important results. The first one, due to Kazhdan, states that a lattice like the one in the theorem is finitely generated and admits no non trivial homomorphism into $\mathbf{R}$ (see [48]). The second, due to Thurston, states that if a finitely generated group Γ has no non trivial homomorphism to $\mathbf{R}$ then any homomorphism from Γ to the group of germs of C^1-diffeomorphisms of $\mathbf{R}$ in the neighbourhood of the fixed point 0 is trivial (see [66]).

If we add more smoothness assumptions (but this is not the goal of this paper...), A. Navas, following earlier ideas of Segal and Reznikov, recently proved a remarkable theorem which applies to groups with Kazhdan's property (T) (see [57]). Note that lattices in higher rank semi-simple Lie groups have this property (see [32]).

THEOREM 7.7 (Navas). *Let Γ be a finitely generated subgroup of the group of diffeomorphisms of the circle of class $C^{1+\alpha}$ with $\alpha > 1/2$. If Γ satisfies Kazhdan's property (T), then Γ is finite.*

When the Lie group G is not simple but only semi-simple, the situation is more complicated since there are some interesting examples of irreducible higher rank lattices that do act. We have already described some examples of irreducible lattices in $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ which act on the circle via their projection on the first factor (which is a dense subgroup in $\mathrm{SL}(2,\mathbf{R})$). As a matter of fact, the next result shows that these examples are basically the only ones.

If ϕ_1 and $\phi_2 \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ are homomorphisms, we say that ϕ_1 is *semi-conjugate to a finite cover of* ϕ_2 if there is a continuous map $h\colon \mathbf{S}^1 \to \mathbf{S}^1$ which is onto and locally monotonous, such that for every $\gamma \in \Gamma$ we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

THEOREM 7.8 ([26]). *Let Γ be an irreducible lattice in a semi-simple Lie group G with real rank greater than or equal to 2. Let ϕ be a homomorphism from Γ to the group of orientation preserving homeomorphisms of the circle. Then either $\phi(\Gamma)$ has a finite orbit or ϕ is semi-conjugate to a finite cover of a homomorphism which is the composition of:*

i) *the embedding of Γ in G,*

ii) *a surjection from G to* $\mathrm{PSL}(2,\mathbf{R})$,

iii) *the projective action of* $\mathrm{PSL}(2,\mathbf{R})$ *on the circle.*

These theorems show that higher rank lattices have very few actions on the circle. Hence, according to Section 6.15, the second bounded cohomology groups of lattices should be small. This is indeed what Burger and Monod showed in [12]:

THEOREM 7.9 (Burger, Monod). *Let Γ be a cocompact irreducible lattice in a semi-simple Lie group G with real rank greater than or equal to 2. Then the second bounded cohomology group $H_b^2(\Gamma,\mathbf{R})$ injects in the usual cohomology group $H^2(\Gamma,\mathbf{R})$.*

The assumption that the lattice is cocompact is important in the proof but the theorem probably generalizes to non-cocompact lattices. Note also that for many lattices in semi-simple Lie groups, it turns out that the usual cohomology group $H^2(\Gamma, \mathbf{R})$ vanishes. This is the case for instance for cocompact torsion free lattices in $\mathrm{SL}(n, \mathbf{R})$ for $n \geq 4$ but more generally for cocompact torsion free lattices in the group of isometries of an irreducible symmetric space of non compact type of rank at least 3 which is not hermitian symmetric (see [7]). In these cases, Theorem 7.9 means that $H_b^2(\Gamma, \mathbf{R})$ vanishes. Hence, using 6.6, we deduce that every action Γ on the circle has a finite orbit. In other words, Theorems 7.4 and 7.9 are closely related and, indeed they have been proved simultaneously (and independently). It would be very useful to compare the two proofs.

As we have already noticed, the vanishing of the second bounded cohomology group is closely related to the notion of commutator length. If Γ is any group and γ is in the first commutator subgroup Γ', we denote by $|\gamma|$ the least integer k such that γ can be written as a product of k commutators. We "stabilize" this number and define $||\gamma||$ as $\lim_{n\to\infty} |\gamma^n|/n$ (which always exists by sub-additivity). It turns out that for a finitely generated group Γ it is equivalent to say that the second bounded cohomology group $H_b^2(\Gamma, \mathbf{R})$ injects in the usual cohomology group $H^2(\Gamma, \mathbf{R})$, and to say that this "stable commutator norm" $|| \ ||$ vanishes identically [5]. Theorem 7.9 therefore implies that for cocompact higher rank lattices, this stable norm vanishes. The following question is natural:

PROBLEM 7.10. *Let Γ be an irreducible lattice as in Theorem 7.4. Does there exist an integer $k \geq 1$ such that every element of the first commutator subgroup of Γ is a product of k commutators ?*

Recall that by a theorem of Kazhdan, there is no non trivial homomorphism from Γ to $\mathbf{R}$; this is equivalent to the fact that the first commutator group of Γ has finite index in Γ. A positive answer to the previous question would be a strengthening of this fact.

7.3 LATTICES IN LINEAR GROUPS

In this section, we prove Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 3)$. The general case of a semi-simple Lie group is much harder but the proof that we present here contains the main ideas. As a matter of fact, we shall first concentrate on the case of a lattice Γ in $\mathrm{SL}(3,\mathbf{R})$ and we shall easily deduce the general case of $\mathrm{SL}(n,\mathbf{R})$ later.

Let us first informally describe the structure of the proof. Let Γ be a lattice in $\mathrm{SL}(3,\mathbf{R})$ and consider a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$.

FIRST STEP. In order to prove the theorem, it is enough to show that there is a probability measure μ on the circle which is invariant under the group $\phi(\Gamma)$.

SECOND STEP (CLASSICAL). A *flag* in $\mathbf{R}^3$ is a pair consisting of a 2-dimensional (vector) subspace E_2 in $\mathbf{R}^3$ and a 1-dimensional (vector) subspace E_1 contained in E_2. Those flags, equipped with the natural topology, define a compact manifold Fl which is a homogeneous space under the action of $\mathrm{SL}(3,\mathbf{R})$. Note that in particular, Γ acts on Fl.

Let $Prob(\mathbf{S}^1)$ be the space of all probability measures on the circle. Equipped with the weak topology, this is a compact metrizable space on which the group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts naturally. The lattice Γ also acts on $Prob(\mathbf{S}^1)$ via the homomorphism ϕ.

Equip Fl with the σ-algebra of Lebesgue measurable sets and $Prob(\mathbf{S}^1)$ with the σ-algebra of Borel sets. In the second step, we construct a measurable map $\Psi\colon Fl \to Prob(\mathbf{S}^1)$ which is equivariant with respect to the actions of Γ on Fl and $Prob(\mathbf{S}^1)$.

In order to prove the theorem, if is enough to show that this map Ψ takes the same value μ almost everywhere with respect to the Lebesgue measure on Fl. Indeed, by equivariance, this measure μ will be invariant by the group $\phi(\Gamma)$.

By way of contradiction, we now assume that Ψ is not constant on a set of full Lebesgue measure.

THIRD STEP. Using ergodic properties of the action of Γ on Fl, we show that there is an integer k and a measurable map Ψ as above such that the image of almost every flag in Fl is the sum of k Dirac masses on the circle (each with weight $1/k$). Let us denote by $\mathbf{S}^1_k$ the set of subsets of $\mathbf{S}^1$ with k elements so that we can now consider Ψ as a map from Fl to $\mathbf{S}^1_k$.

FOURTH STEP. Let X be the space consisting of triples (E_2^1, E_2^2, E_2^3) of distinct planes in $\mathbf{R}^3$ intersecting on the same line E_1. This is again a homogeneous space under the action of $\mathrm{SL}(3,\mathbf{R})$. An element of X determines three flags. Therefore the map Ψ enables us to define a measurable map $\Psi^{(3)}: X \to (\mathbf{S}^1_k)^3$. We will get a contradiction between the ergodicity of the action of Γ on X and the non ergodicity of the action of Γ on the set of triples of points: a triple of points on $\mathbf{S}^1$ can be positively or negatively ordered on the circle and this is invariant under $\mathrm{Homeo}_+(\mathbf{S}^1)$.

We now give the detailed proof.

FIRST STEP: FINDING AN INVARIANT MEASURE. Suppose that there is a probability measure μ on the circle which is invariant under $\phi(\Gamma)$.

We know that the rotation number mapping ρ: $\mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}/\mathbf{Z}$ is not a homomorphism. However by 6.18, the restriction to the subgroup consisting of homeomorphisms preserving a given measure μ is a homomorphism. It follows that the map $\gamma \in \Gamma \mapsto \rho(\phi(\gamma)) \in \mathbf{R}/\mathbf{Z}$ is a homomorphism. According to the result of Kazhdan that we mentioned several times already, Γ is finitely generated and every homomorphism from Γ to $\mathbf{R}$ is trivial. It follows that the image of the restriction of ρ to Γ is a finite cyclic subgroup $\mathbf{Z}/k\mathbf{Z}$. Consider the kernel Γ_0 of this homomorphism: this is a subgroup of index k of Γ, hence a lattice in $\mathrm{SL}(3,\mathbf{R})$. We claim that the support of μ is fixed pointwise by Γ_0. This follows from the fact that for every homeomorphism of the circle with zero rotation number, the support of every invariant measure is contained in the set of fixed points. Hence every point in the support of μ has a finite orbit under $\phi(\Gamma)$. This is the conclusion of Theorem 7.4.

SECOND STEP: FURSTENBERG MAP. This step is classical in the study of actions of lattices and is due to Furstenberg [23].

PROPOSITION 7.11. *There is a Lebesgue measurable map $\Psi: Fl \to Prob(\mathbf{S}^1)$ which is equivariant under the actions of Γ on Fl and $Prob(\mathbf{S}^1)$.*

Proof. We observed that Fl is homogeneous under the action of $\mathrm{SL}(3,\mathbf{R})$. The stabilizer of the flag consisting of the line spanned by $(1,0,0)$ and the plane generated by $(1,0,0)$ and $(0,1,0)$ is the group B of upper triangular matrices. Therefore we can identify Fl and the homogeneous space $\mathrm{SL}(3,\mathbf{R})/B$.

Note that the group B is solvable. Hence B is amenable and there is a linear form m on $L^\infty(B,\mathbf{R})$ which is non negative on non negative functions, takes the value 1 on the constant function 1 and is invariant under left translations. It turns out that it is possible to choose m in such a way that it is a measurable function (see [55]). In other words, if $f_\lambda \in L^\infty(B, \mathbf{R})$ depends measurably on a parameter λ in $[0,1]$, the function $\lambda \mapsto m(f_\lambda)$ is Lebesgue measurable.

Coming back to our problem, we first observe that there are measurable maps $\Psi_0 \colon \mathrm{SL}(3,\mathbf{R}) \to Prob(\mathbf{S}^1)$ which are Γ-equivariant. This follows from the fact that the action of Γ on $\mathrm{SL}(3,\mathbf{R})$ by left translations has a fundamental domain; we define Ψ_0 in an arbitrary measurable way on this fundamental domain and we can therefore define it everywhere using the equivariance.

To complete the proof of the proposition, we modify Ψ_0 to make it invariant under right translations under B. Of course, we use the mean m. We define $\Psi \colon \mathrm{SL}(3,\mathbf{R}) \to Prob(\mathbf{S}^1)$ in the following way. If $g \in \mathrm{SL}(3,\mathbf{R})$, the probability $\Psi(g)$ is defined by its value on a continuous function $u \colon \mathbf{S}^1 \to \mathbf{R}$:

$$\int_{\mathbf{S}^1} u\, d\Psi(g) = m(x \in B \mapsto \int_{\mathbf{S}^1} u\, d\Psi_0(gx))\,.$$

By construction, Ψ is measurable and invariant by right translations by B; this defines another measurable map $\Psi \colon Fl \simeq \mathrm{SL}(3,\mathbf{R})/B \to Prob(\mathbf{S}^1)$ which is Γ-equivariant, as required. □

THIRD STEP: THE MAP Ψ TO DIRAC MASSES. As mentioned above, *we now assume by contradiction that the map Ψ is not constant on a subset of full Lebesgue measure.*

PROPOSITION 7.12. *There exist an integer $k \geq 1$ and a map $\Psi \colon Fl \to \mathbf{S}^1_k$ to the set of subsets of $\mathbf{S}^1$ with k elements which is Lebesgue measurable and Γ-invariant.*

In order to prove the proposition, we first recall an important ergodic theorem due to Moore that we shall use repeatedly (see for instance [72]). Let $Y = G/H$ be a homogeneous space of a semi-simple Lie group G. Assume that G is connected, has a finite center and has no compact factor. Assume moreover that H is non compact. Let Γ be an irreducible lattice in G. Then the action of Γ on Y is ergodic with respect to the Lebesgue measure (class), *i.e.* every measurable function on Y which is Γ-invariant is constant almost everywhere.

For instance, the stabilizer B of a flag is non compact. *The action of Γ on Fl is ergodic.*

As another example, let us consider the space Y of pairs of flags of $\mathbf{R}^3$ which are in general position. For such a pair of flags, there are three non coplanar lines E_1^1, E_1^2, E_1^3 such that the first flag is given by the line E_1^1 and the plane spanned by E_1^1 and E_1^2 and the second flag is given by the line E_1^3 and the plane spanned by E_1^2 and E_1^3. Since $\mathrm{SL}(3,\mathbf{R})$ acts transitively on the space of triples of non coplanar lines, it follows that Y is a homogeneous space of $\mathrm{SL}(3,\mathbf{R})$. The stabilizer of an element of Y is the stabilizer of a triple of non coplanar lines: it is clearly non compact. Consequently, the action of Γ on Y is ergodic. Since the set of pairs of flags in general position has full Lebesgue measure in the set of pairs of flags, we deduce that Γ acts ergodically on the set of pairs of flags of $\mathbf{R}^3$.

However, the reader will easily check that this cannot be generalized to the set of triples of flags: the action of $\mathrm{SL}(3,\mathbf{R})$ is not transitive on the set of triples of flags in general position.

In order to prove Proposition 7.12, we analyze the action of Γ on the space of pairs of probability measures on the circle.

If μ is a probability on the circle, we define *atom*(μ) as the sum of the masses of the atoms of μ (*i.e.* those points x such that $\mu(\{x\}) > 0$). This is a measurable function on *Prob*$(\mathbf{S}^1)$ which is invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The map:

$$d \in Fl \mapsto atom(\Psi(d)) \in [0,1]$$

is a measurable Γ-invariant function. Using the ergodicity result that we mentioned above, this function is constant almost everywhere.

Assume first that this constant is not zero. This means that the image of almost every flag under Ψ has at least one atom.

Let $\alpha > 0$ be a positive real number. For each probability measure μ on the circle, consider the points x such that $\mu(\{x\}) > \alpha$. Of course, the number of those points x is finite (possibly zero). Denote this number by $N(\mu,\alpha)$. The map $d \in Fl \mapsto N(\Psi(d),\alpha) \in \mathbf{N}$ is measurable and Γ-invariant; it is therefore constant, equal to some integer N_α almost everywhere. Since we assume that for almost every d the probability $\Psi(d)$ has at least one atom, we can choose some α in such a way that N_α is an integer $k \geq 1$. This enables us to construct a map (defined almost everywhere) from Fl to the set of subsets of $\mathbf{S}^1$ with k elements, sending the flag d to the k atoms of $\Psi(d)$ having a mass greater than or equal to α. Changing our notation, we shall call this new map Ψ: it satisfies Proposition 7.12 which is therefore proved, if almost every $\Psi(d)$ has at least one atom.

We now assume that for almost every d, the probability $\Psi(d)$ has no atom.

We shall show that under this assumption, almost all the measures $\Psi(d)$ have the same support.

Let μ_1 and μ_2 be two probability measures on the circle with no atom. Define $D(\mu_1, \mu_2)$ as the maximum of the μ_2-measures of the connected components of the complement of the support of μ_1. If $D(\mu_1, \mu_2) = 0$, the support of μ_1 contains the support of μ_2. The map

$$(d_1, d_2) \in Fl^2 \mapsto D(\Psi(d_1), \Psi(d_2)) \in [0, 1]$$

is defined almost everywhere and is Γ-invariant. Using the same ergodicity result as before, we deduce that it is constant almost everywhere. *We claim that this constant δ is 0.*

Suppose on the contrary that $\delta > 0$. Using Fubini's theorem, we can find a measurable part $\Omega \subset Fl$ such that:

- Ω has full Lebesgue measure.
- If $d \in \Omega$, the probability $\Psi(d)$ has no atom.
- If $d \in \Omega$, then $D(d, d') = \delta$ for almost every d' in Fl.
- If $d \in \Omega$, then $\Psi(d)$ belongs to the support of the measure $\Psi_*(Lebesgue)$ on the compact metrizable space $Prob(\mathbf{S}^1)$.

Fix a point $d \in \Omega$. We can find a sequence $d_i \in \Omega$ such that $\Psi(d_i) = \mu_i$ converges towards $\Psi(d) = \mu$. The probability measures μ_i have no atoms and $D(\mu_i, \mu) = \delta$. This means that there is a component I_i of the complement of the support $supp(\mu)$ such that $\mu_i(I_i) = \delta$. If the sequence of lengths of I_i converges to 0, we can assume that the sequence of intervals I_i shrinks to a point p. This implies that the point p is an atom of μ, contradicting our assumption. Therefore we can assume (after taking a subsequence) that the intervals I_i all coincide with some interval I. Since we know that the endpoints of I are not atoms of μ, that the sequence μ_i converges weakly to μ, and that $\mu_i(I) = \delta$, it follows that $\mu(I) = \delta$. This contradicts the fact that I is in the complement of the support of μ.

We showed that $\delta = 0$. This means that for almost every pair of flags (d, d'), we have $D(\Psi(d), \Psi(d')) = 0$. Therefore, for almost every pair of flags (d, d'), the probability measures $\Psi(d)$ and $\Psi(d')$ have the same support. In other words, *there exists a compact set $K \subset \mathbf{S}^1$ with no isolated point, such that for almost every flag d, the support of $\Psi(d)$ is equal to K.*

Each connected component of $\mathbf{S}^1 - K$ is an open interval. Collapsing the closure of these intervals to a point, we get a space homeomorphic to a circle. Therefore, there exists a continuous $\pi \colon \mathbf{S}^1 \to \mathbf{S}^1$ such that each fiber of π is a point or the closure of a component of the complement of K. If μ is a measure with no atom whose support is K, the direct image $\pi_\star(\mu)$ is a measure on the circle with no atom and full support on the circle.

Using $\pi_\star$, we get a map $\overline{\Psi}$ from Fl to the space of probability measures on the circle with no atom and full support which is Γ-equivariant with respect to the minimal action $\overline{\phi}$ associated to ϕ (see 5.8).

The space of probability measures with no atoms and full support on the circle is a homogeneous space under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and the stabilizer of the Lebesgue measure is of course $\mathrm{SO}(2)$. This space can therefore be identified with the quotient $\mathrm{Homeo}_+(\mathbf{S}^1)/\,\mathrm{SO}(2)$. The group $\mathrm{Homeo}_+(\mathbf{S}^1)$, as any metrizable topological group, can be equipped with a left invariant metric, that we can average under the action of $\mathrm{SO}(2)$ to produce a left invariant metric *dist* on $\mathrm{Homeo}_+(\mathbf{S}^1)/\,\mathrm{SO}(2)$. In practice, we could simply define $dist(\mu_1,\mu_2)$ as the supremum of $|\mu_1(I)-\mu_2(I)|$ where I runs through the collection of intervals on the circle: it is easy to check that this metric indeed defines the weak topology when restricted to the set of probability measures with no atom and full support.

For almost every pair of flags (d,d') the distance $dist(\overline{\Psi}(d),\overline{\Psi}(d'))$ defines a Γ-invariant function of pairs of flags; it is therefore constant almost everywhere. Using the same argument as above, we see that this constant is 0, which means that the map $\overline{\Psi}$ is constant almost everywhere. Of course, two probability measures with no atom and with support in K which have the same image under $\pi_\star$ have to coincide so that we deduce that Ψ is constant almost everywhere. We have found a probability measure on the circle which is invariant under $\phi(\Gamma)$. This is a contradiction with our initial assumption and proves 7.12.

FOURTH STEP: CYCLIC ORDERING ON TRIPLES OF POINTS ON A CIRCLE. In order to explain the general idea, we assume first that the integer k that we introduced is equal to 1. In other words, we have a Γ-invariant map $\Psi\colon Fl \to \mathbf{S}^1$ defined almost everywhere which is not constant on a set of full Lebesgue measure.

As explained above, let X denote the space of triples (E_2^1, E_2^2, E_2^3) of distinct planes in $\mathbf{R}^3$ intersecting on the same line E_1. This is again a homogeneous space under $\mathrm{SL}(3,\mathbf{R})$ and the stabilizer of a point in X is clearly non compact. We deduce from Moore ergodicity theorem that the action of Γ on X is ergodic. Since a point of X determines three flags, we can define a measurable Γ-equivariant map $\Psi^{(3)}\colon X \to (\mathbf{S}^1)^3$ (defined almost everywhere). Indeed, let us consider the projection $pr\colon Fl \to \mathbf{RP}^2$ from Fl to the real projective plane mapping a flag $E_1 \subset E_2$ to the line $E_1 \subset \mathbf{R}^3$. The space X

is therefore the space of triples of flags having the same projection under pr. It follows from Fubini's theorem that for every subset of full measure in Fl, the set of triples of elements of this set having the same projection under pr has full measure in X: this is exactly what we need to define $\Psi^{(3)}$.

The space $(\mathbf{S}^1)^3$ can be decomposed into disjoint parts, invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$:

i) Triples of the form (x,x,x).

ii) Triples consisting of two distinct points. In turn, this set can be decomposed into three parts: the spaces of triples of the form (x,x,z), resp. (x,y,x), resp. (x,y,y).

iii) Triples (x,y,z) of distinct elements on the circle whose cyclic ordering is positive, *i.e.* such that the interval positively oriented from x to y does not contain z.

iv) Triples (x,y,z) of distinct elements on the circle whose cyclic ordering is negative.

Inverse images of these six parts under $\Psi^{(3)}$ are measurable and disjoint Γ-invariant sets and therefore have to be either of measure 0 or of full Lebesgue measure. This means that there is a subset $\Omega \subset X$ of full measure whose image is contained in one of the six parts that we described. We claim that this is not possible.

Observe that the symmetric group $\mathfrak{S}_3$ of permutations of three objects acts on X and on $(\mathbf{S}^1)^3$, permuting respectively flags and points. Note that these actions commute with the actions of Γ on X and $(\mathbf{S}^1)^3$. Of course $\Psi^{(3)}$ is equivariant with respect to these action of $\mathfrak{S}_3$.

It follows that the part which contains $\Psi^{(3)}(\Omega)$ has to be invariant under $\mathfrak{S}_3$. Among the 6 parts that we described, only the first one has this property. This means that the map $\Psi\colon Fl \to \mathbf{S}^1$ factors through the projection $pr\colon Fl \to \mathbf{RP}^2$. In other words, almost everywhere, the image of a flag by Ψ depends only on the line associated to the flag and not on its plane.

Exactly in the same way, we could have defined a space X' consisting of triples of flags having the same plane, *i.e.* having the same projection in the dual projective plane. The same proof shows that almost everywhere Ψ depends only on the plane of a flag and not on its line.

This implies that Ψ is constant almost everywhere and gives the contradiction we were looking for when $k=1$.

When $k>1$, we shall use a similar idea.

Recall that we denote by $\mathbf{S}^1_k$ the space of subsets A of the circle with k elements. Given two elements (A_1,A_2,A_3) and (A'_1,A'_2,A'_3) of $(\mathbf{S}^1_k)^3$, we say that they have the same cyclic ordering if there is an orientation preserving homeomorphism h of the circle such that $h(A_1) = A'_1, h(A_2) = A'_2, h(A_3) = A'_3$. This gives a partition of $(\mathbf{S}^1_k)^3$ into finitely many parts invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$. As before, it follows that there is a subset Ω of full measure in X such that $\Psi(\Omega)$ is contained in one of these subsets. Using again the action of $\mathfrak{S}_3$ we conclude that this subset consists of triples (A_1, A_2, A_3) which have the same cyclic ordering as $(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)})$ for every element $\sigma \in \Sigma_3$. Therefore, for every σ, there is an orientation preserving homeomorphism h_σ such that $h_\sigma(A_i) = A_{\sigma(i)}$ for $i = 1, 2, 3$. Let A be the union of A_1, A_2 et A_3: this is a set with $N \leq 3k$ elements. Orientation preserving homeomorphisms globally preserving A must induce a cyclic permutation of its elements. In particular, the commutator of two elements h_σ must fix each element of A since cyclic permutations commute. As the cyclic permutation $\sigma = (1,2,3)$ is a commutator in $\mathfrak{S}_3$, the homeomorphism $h_{(1,2,3)}$ acts trivially on A. Since we know that $h_{(1,2,3)}(A_1) = A_2$, $h_{(1,2,3)}(A_2) = A_3$ and $h_{(1,2,3)}(A_3) = A_1$, we have $A_1 = A_2 = A_3$. We showed that there exists a measurable subset of full measure $\Omega \subset X$ such that the image $\Psi(\Omega)$ consists of triples of the form (A, A, A). Exactly as we did in the case $k = 1$, we conclude that Ψ is constant almost everywhere and this is a contradiction.

This is the end of the proof of Theorem 7.4 for lattices in $\mathrm{SL}(3, \mathbf{R})$.

Remark that the core of the proof is the incompability between two facts. The group $\mathrm{Homeo}_+(\mathbf{S}^1)$ does not act transitively on generic triples of points on the circle but $\mathrm{SL}(3, \mathbf{R})$ does act transitively on X. Note that the existence of an element of $\mathrm{SL}(3, \mathbf{R})$ fixing a line and permuting arbitrarily three planes containing this line, means that the real projective plane is not orientable.

The proof for a lattice Γ in $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 3)$ is very similar. For every sequence of integers, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, we consider the space $Fl_{i_1,\ldots,i_l}$ of flags of type $(i_1, \ldots, i_l)$, *i.e.* sequences of vector sub-spaces $E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_l} \subset \mathbf{R}^n$ with $\dim E_{i_j} = i_j$ $(j = 1, \ldots, l)$. This is a homogeneous space under the action of $\mathrm{SL}(n, \mathbf{R})$. The space of complete flags, *i.e.* $Fl = Fl_{1,2,\ldots,n}$ is equipped with projections pr_j on incomplete flag spaces $Fl_{1,2,\ldots,\hat{j},\ldots,n}$ where the index j does not appear. The space X_j consisting of distinct triples of flags Fl having the same projection under pr_j is again a homogeneous space of $\mathrm{SL}(n, \mathbf{R})$, with non compact stabilizer.

Now, the proof is the same as before. We first construct an equivariant map Ψ from Fl to $Prob(\mathbf{S}^1)$ (same proof). Assuming by contradiction that Ψ is not constant almost everywhere, we get another map, still denoted by Ψ from Fl to $\mathbf{S}^1_k$ (same proof). For each $j = 1, \ldots, n$, we consider the corresponding map $\Psi_j^{(3)}: X_j \to \mathbf{S}_k^1$ and we show, as above, that the image of this map consists almost everywhere of triples of the form (A, A, A). It follows that for each $j = 1, \ldots, n$ and on a subset of full measure, the image of a flag by Ψ depends only on its projection by pr_j. Since this is true for every j, this means that Ψ is constant almost everywhere. This is a contradiction and finishes the proof of Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$.

Of course, these proofs immediately generalize to lattices in complex or quaternionic special linear groups $\mathrm{SL}(n, \mathbf{C})$ and $\mathrm{SL}(3, \mathbf{H})$ (for $n \geq 3$).

7.4 Some groups that do act...

We saw that many higher rank lattices don't act on the circle. To conclude these notes, we give some more examples of "big" groups acting on the circle. Let Σ be a compact oriented surface of genus $g \geq 2$ and $x \in \Sigma$ be some base point. The fundamental group $\pi_1(\Sigma, x)$ is a classical example of a hyperbolic group in the sense of Gromov (see for instance [27]). The boundary of this group is a topological circle: indeed $\pi_1(\Sigma, x)$ acts freely and cocompactly on the Poincaré disc so that $\pi_1(\Sigma, x)$ is quasi-isometric to the Poincaré disc. Consequently, the automorphism group $\mathrm{Aut}(\pi_1(\Sigma, x))$ acts naturally on the circle. This action is very interesting and has been very much studied. See for instance [21]. Note that $\mathrm{Aut}(\pi_1(\Sigma, x))$ contains the group of inner conjugacies and that the quotient $\mathrm{Out}(\pi_1(\Sigma, x))$ is the *mapping class group* of the surface (*i.e.* the group of isotopy classes of homeomorphisms of the surface):

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma, x)) \longrightarrow \mathrm{Out}(\pi_1(\Sigma, x)) \longrightarrow 1\,.$$

Fix an element f of infinite order in this mapping class group and consider the group Γ_f which is the inverse image of the group generated by f in the previous exact sequence. We have an exact sequence:

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \Gamma_f \longrightarrow \mathbf{Z} \longrightarrow 1\,.$$

This group Γ_f is the fundamental group of the 3-manifold which fibers over the circle and whose monodromy is given by the class f. Thurston showed that if f is of pseudo-Anosov type, then this 3-manifold is hyperbolic. In particular, for such a choice of f, the group Γ_f embeds as a discrete cocompact subgroup of the isometry group of the hyperbolic 3-ball, isomorphic to $\mathrm{PSL}(2, \mathbf{C})$. This construction provides many examples of faithful actions of (rank 1) lattices on the circle. In [68] Thurston constructs faithful actions of the fundamental group of many hyperbolic 3-manifolds on the circle.

Suppose now that Σ has one boundary component $\partial\Sigma$. Choose the base point on the boundary and equip Σ with a metric with curvature -1 and totally geodesic boundary. The universal cover $\widetilde{\Sigma}$ of Σ is therefore identified with the complement in the Poincaré disc of a disjoint union of half spaces. On the boundary of the disc, these half spaces define an open dense subset Ω whose complement is a Cantor set K which is the boundary of the hyperbolic group $\pi_1(\Sigma, x)$. The union $\partial\widetilde{\Sigma} \cup K$ is a topological circle and if we collapse each connected component of $\partial\widetilde{\Sigma}$ to a point, this circle collapses to another circle that we denote by C. Choose also a base point $\widetilde{x}$ above x in the universal cover. Consider now the mapping class group Γ of Σ *i.e.* the group of homeomorphisms of Σ modulo isotopy. A homeomorphism f of Σ has a lift $\widetilde{f}$ to $\widetilde{\Sigma}$ which fixes the boundary component containing $\widetilde{x}$. This homeomorphism $\widetilde{f}$ extends continuously to $\partial\widetilde{\Sigma} \cup K$ and defines a homeomorphism $\bar{f}$ of the circle C. Note that if two homeomorphisms are isotopic, the two corresponding extensions agree on the Cantor set K. The connected component of $\partial\widetilde{\Sigma}$ containing $\widetilde{x}$ yields a base point $\bar{x}$ in C which is fixed by all homeomorphisms $\bar{f}$. Hence we can define an action of Γ on a line by letting f act via $\bar{f}$ on the line $C - \{\bar{x}\}$. Hence we proved (following an idea of Thurston) that the mapping class group of (Σ, x) acts (faithfully) on a line and is therefore left orderable.

We could also use the same idea for surfaces with several boundary components, for instance the sphere minus a finite number of discs. The corresponding mapping class groups turn into the so called braid groups. In this way we get interesting faithful actions of braid groups on the line, or equivalently total left orderings. It is interesting to note that these orderings were initially discovered from a completely different point of view by Dehornoy [16].

To conclude this paper, we would like to mention a rich family of group actions on the circle, coming from the theory of Anosov flows on 3-manifolds. Let M be a compact connected 3-manifold with no boundary and X a non singular smooth vector field on M. Denote by ϕ^t the flow generated by X. One says that ϕ^t is an *Anosov flow* if there is a continuous splitting of the tangent bundle TM as a sum of three line bundles $TM = \mathbf{R}X \oplus E^{ss} \oplus E^{uu}$ which are invariant under (the differential of) the flow ϕ^t and such that vectors in E^{uu} are expanded, and vectors in E^{ss} are contracted. More precisely, this means that for any riemannian metric on M, there are constants $C > 0$ and $\lambda > 0$ such that for any time $t > 0$ and vectors $v_{ss} \in E^{ss}$ and $v_{uu} \in E^{uu}$,

$$||d\phi^t(v_{ss})|| \leq C\exp(\lambda t)||v_{ss}||\,,$$

$$||d\phi^t(v_{uu})|| \geq C\exp(\lambda t)||v_{uu}||\,.$$

This kind of flow is rather abundant on 3-manifolds. The main example, which gave birth to the theory, is the geodesic flow of a compact surface with negative curvature, acting on the unit tangent bundle of the surface. We refer to [3, 22] for a general presentation of the theory including a bibliography. Starting from some Anosov flow and selecting a periodic orbit, one can perform a Dehn surgery on this closed curve. It turns out that if the surgery is positive, one can define a flow on the new manifold which is still of Anosov type. Using this construction, one constructs many examples. For instance, one can construct Anosov flows on some hyperbolic 3-manifolds (*i.e.* admitting a metric of constant negative curvature).

One of the main properties of Anosov flows is that they give rise to two codimension one foliations. Indeed, it has been shown by Anosov that there are two codimension one foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ whose leaves are everywhere tangent to $E^{ss} \oplus \mathbf{R}X$ and $E^{ss} \oplus \mathbf{R}X$. Verjovsky showed that if one lifts the flow ϕ^t to the universal cover $\widetilde{M}$ of M, the orbits of the resulting flow $\widetilde{\phi}^t$ are the fibers of a (trivial) fibration of $\widetilde{M}$ over a surface S (diffeomorphic to $\mathbf{R}^2$). Lifting the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ to $\widetilde{M}$, we get two foliations which project to two transverse foliations by curves $\widetilde{f}^u$ and $\widetilde{f}^s$ on the surface S. One says that the flow is $\mathbf{R}$-*covered* if the leaves of $\widetilde{f}^u$ are the fibers of a (trivial) fibration $p_u \colon S \to \mathbf{R}_u$ (where $\mathbf{R}_u$ is homeomorphic to $\mathbf{R}$). It follows that the leaves of $\widetilde{f}^s$ are also the fibers of a (trivial) fibration $p_s \colon S \to \mathbf{R}_s$. For instance, the geodesic flow on a negatively curved surface is $\mathbf{R}$-covered. It turns out that a positive surgery on an $\mathbf{R}$-covered Anosov flow is still $\mathbf{R}$-covered so that we get many examples. Consider the map $(p_u, p_s) \colon S \to \mathbf{R}_u \times \mathbf{R}_s$. Barbot and Fenley showed independently that this map is bijective if and only if the Anosov flow is the suspension of some Anosov diffeomorphism of the 2-torus. In all other cases, they showed that the image of (p_u, p_s) is an open strip in $\mathbf{R}_u \times \mathbf{R}_s$ of the form $\{(x,y) \mid h_-(x) < y < h_+(x)\}$ where h_- and h_+ are some homeomorphisms from $\mathbf{R}_u$ to $\mathbf{R}_s$. Now, observe that the fundamental group Γ of the manifold M acts on all these objects so that we get in particular actions of Γ on $\mathbf{R}_u$ and $\mathbf{R}_s$ which are conjugate by h_u and h_s. Denote by τ the composition $h_u h_s^{-1}$: this is a homeomorphism of $\mathbf{R}_u$ which acts freely so that we can define a circle $\mathbf{S}^1_u$ by taking the quotient of $\mathbf{R}_u$ by the action of τ. Note that the action of Γ on $\mathbf{R}_u$ obviously commutes with τ so that we get an action of Γ on $\mathbf{S}^1_u$. In case we start with the geodesic flow of a negatively curved surface Σ, the fundamental group Γ is a central extension of the fundamental group $\pi_1(\Sigma)$ by $\mathbf{Z}$. The action of Γ that we get on $\mathbf{S}^1_u$ is not faithful: the center $\mathbf{Z}$ acts trivially and the induced action of $\pi_1(\Sigma)$ on the circle is of course the familiar projective action. If the $\mathbf{R}$-covered Anosov flow is not the geodesic flow (up to a finite cover), the action of Γ on $\mathbf{S}^1_u$ is faithful. For instance, we get in this way some examples of faithful actions of the fundamental group of some hyperbolic 3-manifolds on the circle.

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