Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GROUPS ACTING ON THE CIRCLE

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Kapitel: 6.5 Actions on the real line and orderings

DOI: https://doi.org/10.5169/seals-65441

have zero norm... Consider the case of the bounded Euler class, seen in the real bounded cohomology.

THEOREM 6.7. *The image of the bounded Euler class eu in the real bounded cohomology* $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ *has norm* $1/2$.

Proof. This is the abstract version of the Milnor-Wood inequality. Note that a constant 2-cocycle is the coboundary of a constant 1-cochain. We found a representative of the Euler class taking only two values 0 and 1. If we subtract from this cocycle the constant cocycle taking the value $1/2$, we get a cohomologous bounded (real) cocycle taking values $\pm 1/2$. This shows that the norm of the image of eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ is at most $1/2$. The opposite inequality follows from Milnor's computation of the Euler number for an embedding of the fundamental group Γ_g of a closed oriented surface as a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that we mentioned in 6.1. If the norm were strictly less than $1/2$, then this number would be strictly less than $2g - 2$. See [25] for more explanations. □

6.5 ACTIONS ON THE REAL LINE AND ORDERINGS

Our main concern is to study actions on the circle but there is a preliminary question which deals with actions on the line. Of course, if a group acts on the line, we can always add a point at infinity to produce an action on the circle (with a common fixed point). In other words studying actions on the line is equivalent to studying actions on the circle with vanishing bounded Euler class. This is the reason why we begin by general remarks on groups acting on the line.

Observe first that the dynamics of a single orientation preserving homeomorphism h of $\mathbf{R}$ are very easy to describe. Let $F = Fix(h)$ be the set of fixed points. Each interval of the complement of F is h-invariant and the action of h on this interval is conjugate to a translation (positive or negative, according to the sign of $h(x) - x$ on this interval).

We say that a group Γ is *left orderable* if there exists a *total* ordering $\preceq$ on Γ which is invariant under left translations (*i.e.* $\gamma_1 \preceq \gamma_2$ implies $\gamma\gamma_1 \preceq \gamma\gamma_2$). We write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. An obvious necessary condition for a group to be left orderable is that it be torsion free (*i.e.* there is no non trivial element of finite order).

The following theorem is well known but we weren't able to find its origin in the literature.

THEOREM 6.8. *Let Γ be a countable group. Then the following are equivalent:*

1) *Γ acts faithfully on the real line by orientation preserving homeomorphisms.*

2) *Γ is left orderable.*

Proof. Suppose that Γ acts faithfully on the line by orientation preserving homeomorphisms, *i.e.* that there exists an injective homomorphism ϕ from Γ into the group $\mathrm{Homeo}_+(\mathbf{R})$ of orientation preserving homeomorphisms of the real line. Assume first that there is a point x_0 in $\mathbf{R}$ with trivial stabilizer. Then we can define a left invariant total ordering by defining $\gamma_1 \preceq \gamma_2$ if $\phi(\gamma_1)(x_0) \leq \phi(\gamma_2)(x_0)$. If there is no such point x_0, choose a sequence of points $(x_i)_{i\in\mathbf{N}}$ which is dense in the line. Now define $\gamma_1 \preceq \gamma_2$ if $\gamma_1 = \gamma_2$ or if the first i for which $\phi(\gamma_1)(x_i) \neq \phi(\gamma_2)(x_i)$ is such that $\phi(\gamma_1)(x_i) < \phi(\gamma_2)(x_i)$. This defines a left invariant total order on Γ.

Conversely, let $\preceq$ be a left invariant total order on the countable group Γ. Enumerate the elements of Γ, *i.e.*, choose a bijection $i \in \mathbf{N} \mapsto \gamma_i \in \Gamma$. We are going to construct inductively an increasing injection v of $(\Gamma, \preceq)$ in $(\mathbf{R}, \leq)$. Define $v(\gamma_0)$ arbitrarily and suppose that $v(\gamma_0), \dots, v(\gamma_i)$ have been defined. If γ_{i+1} is smaller (resp. bigger) than all $\gamma_0, \dots, \gamma_i$ then define $v(\gamma_{i+1})$ as any real number smaller (resp. bigger) than $\min(v(\gamma_0), \dots, v(\gamma_i)) - 1$ (resp. $\max(v(\gamma_0), \dots, v(\gamma_i)) + 1$). Otherwise, there is a pair of integers $0 \leq \alpha, \beta \leq i$ such that $\gamma_\alpha \prec \gamma_{i+1} \prec \gamma_\beta$ and such that there is no γ_j $(0 \leq j \leq i)$ between γ_α and γ_β. Then we define $v(\gamma_{i+1})$ as $(v(\gamma_\alpha) + v(\gamma_\beta))/2$. Let $\bar{X} \subset \mathbf{R}$ be the closure of $v(\Gamma)$.

By our construction, it is easy to verify that $\bar{X}$ is unbounded and that any connected component $]a,b[$ of the complement of $\bar{X}$ is such that a and b are in $v(\Gamma)$. The group Γ acts on itself by left translations so that every element γ of Γ induces an increasing bijection $\phi(\gamma)$ of $v(\Gamma)$. We claim that $\phi(\gamma)$ extends continuously to $\bar{X}$. Otherwise, there would exist a point $x = \lim_n v(\gamma_{i_n}) = \lim_m v(\gamma_{i_m})$ for an increasing sequence of elements γ_{i_n} and a decreasing sequence γ_{i_m} and such that $\lim_n v(\gamma\gamma_{i_n}) < \lim_m v(\gamma\gamma_{i_m})$. Then $a = \lim_n v(\gamma\gamma_{i_n})$ and $b = \lim_m v(\gamma\gamma_{i_m})$ would be the endpoints of some connected component of the complement of $\bar{X}$. By our previous observation, a and b would be the image by v of two distinct elements of Γ. On multiplying these two elements on the left by γ^{-1}, this would produce two distinct elements γ_- and γ_+ such that $v(\gamma_{i_n}) \leq v(\gamma_-) < v(\gamma_+) \leq v(\gamma_{i_m})$ and this contradicts the fact that the two sequences have the same limit x.

Therefore we have produced a homeomorphism $\phi(\gamma)$ of $\bar{X}$. We now extend $\phi(\gamma)$ to the whole line $\mathbf{R}$ in such a way that $\phi(\gamma)$ is affine on each interval of the complement of $\bar{X}$. It is now clear that ϕ is an injective homomorphism from Γ to the group of orientation preserving homeomorphisms of the real line. $\square$

Theorem 6.8 produces many examples of actions on the real line. For instance, suppose Γ is a countable group containing a nested sequence of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_i \supset \ldots$ (finite or infinite) such that the intersection of this family reduces to the trivial element and that each Γ_i is a normal subgroup in the previous one Γ_{i-1}. Assume that each quotient $Q_i = \Gamma_i/\Gamma_{i-1}$ is left orderable and denote by $\preceq_i$ such a left order on Q_i. Let us construct a left order $\preceq$ on Γ. Consider two distinct elements γ, γ' in Γ and let i be the first integer such that $\gamma\gamma'^{-1}$ is not in Γ_i. Then $\gamma^{-1}\gamma'$ is in Γ_{i-1} and determines an element $[\gamma^{-1}\gamma']$ of Q_i. Then define $\gamma \preceq \gamma'$ if $[\gamma^{-1}\gamma'] \preceq_i 1$. This is a left invariant total order on Γ.

As an example, note that a countable torsion free abelian group A embeds in the tensor product $A \otimes \mathbf{Q}$ which is a $\mathbf{Q}$-vector space whose dimension is at most countable and therefore embeds in $\mathbf{R}$. Hence, countable torsion free abelian groups are orderable. Let us say that a group Γ is solvable (resp. residually solvable) if there is a finite (resp. infinite) decreasing sequence of subgroups as in the previous paragraph such that the quotient groups Q_i are abelian. We have now proved:

PROPOSITION 6.9. *Let Γ be a countable group which is (residually) solvable with torsion free abelian quotients. Then Γ acts faithfully on the real line by orientation preserving homeomorphisms.*

There are many examples of such groups: free groups or fundamental groups of closed orientable surfaces for instance have these properties [46]. Observe that the left orderings that we produced by the previous argument are in fact left *and right* invariant orderings. If we go back to the proof of Theorem 6.8 we can check that for bi-invariant ordered groups, the actions on the line $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ produced by the proof are very peculiar: they are such that for every non trivial $\gamma \in \Gamma$, we have either $\phi(\gamma)(x) \leq x$ for all $x \in \mathbf{R}$ or $\phi(\gamma)(x) \geq x$ for all x. In other words the graphs of $\phi(\gamma)$ don't cross the diagonal. However, there will be elements whose graphs touch the diagonals, unless of course the action is free, which is almost never the case because of the following well known theorem of Hölder.

THEOREM 6.10 (Hölder). *If a group acts freely on the real line by homeomorphisms, it is abelian. More precisely, such a group embeds as a subgroup of* $\mathbf{R}$ *and the action is semi-conjugate to a group of translations. In the same way, a group acting freely on the circle is abelian, embeds in* $\mathrm{SO}(2)$*, and is semi-conjugate to a group of rotations.*

Proof. Let $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ be a homomorphism such that for all γ different from the identity the homeomorphism $\phi(\gamma)$ has no fixed point. If γ, γ' are elements of Γ, write $\gamma \preceq \gamma'$ if $\phi(\gamma)(0) \leq \phi(\gamma')(0)$ (which implies $\phi(\gamma)(x) \leq \phi(\gamma')(x)$ for all x since the action is free). This defines a left and right invariant ordering $\preceq$ which is archimedean, *i.e.* such that for any pair of non trivial elements γ, γ' for which $id \prec \gamma$ and $id \prec \gamma'$, there is a positive integer n such that $\gamma' \prec \gamma^n$. Indeed, the sequence $\phi(\gamma)^n(0)$ is increasing and has to tend to ∞ since otherwise its limit would be a fixed point of $\phi(\gamma)$; hence for n sufficiently large we have $\phi(\gamma')(0) \leq \phi(\gamma^n)(0)$.

Then we show that any group Γ equipped with a bi-invariant total archimedean ordering embeds in $\mathbf{R}$ and is therefore abelian. Fix a non trivial element γ_0 such that $id \prec \gamma_0$ and for each $\gamma \in \Gamma$, define $\Phi(\gamma)$ as the smallest integer $k \in \mathbf{Z}$ such that $\gamma \preceq \gamma_0^k$. We have

$$\gamma_0^{\Phi(\gamma)-1} \prec \gamma \preceq \gamma_0^{\Phi(\gamma)}.$$

This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$ which satisfies

$$\Phi(\gamma) + \Phi(\gamma') - 1 < \Phi(\gamma\gamma') \leq \Phi(\gamma) + \Phi(\gamma')$$

so that Φ is a quasi-homomorphism. As we have already observed, $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a quasi-homomorphism $\phi\colon \Gamma \to \mathbf{R}$ which is homogeneous (*i.e.* $\phi(\gamma^n) = n\phi(\gamma)$) and which is increasing (*i.e.* $\gamma \preceq \gamma'$ implies $\phi(\gamma) \leq \phi(\gamma')$). Note that $\phi(\gamma_0) = 1$.

We claim that ϕ is a group homomorphism. Indeed, consider two elements γ, γ' in Γ and assume for instance that $\gamma\gamma' \preceq \gamma'\gamma$. It follows easily by induction that for every positive integer n, we have $\gamma^n\gamma'^n \preceq (\gamma\gamma')^n \preceq \gamma'^n\gamma^n$. Evaluating Φ on this inequality, we get

$$\Phi(\gamma^n) + \Phi(\gamma'^n) - 1 \leq \Phi((\gamma\gamma')^n) \leq \Phi(\gamma^n) + \Phi(\gamma'^n).$$

Dividing by n and taking the limit, we obtain

$$\phi(\gamma) + \phi(\gamma') \leq \phi(\gamma\gamma') \leq \phi(\gamma) + \phi(\gamma')$$

so that ϕ is indeed a homomorphism.

We still have to show that ϕ is injective. For any γ such that $id \prec \gamma$ we know, since the ordering is archimedean, that there is some positive integer k

such that $\gamma_0 \preceq \gamma^k$. It follows that $1 \leq k\phi(\gamma)$ so that $\phi(\gamma)$ is non trivial. This proves the injectivity of ϕ.

Observe that the non decreasing embedding ϕ of Γ in $\mathbf{R}$ is unique up to a multiplicative constant. Indeed, if ϕ' is another one, we have by definition $(\Phi(\gamma^n)-1)\phi'(\gamma_0) \leq \phi'(\gamma^n) \leq \Phi(\gamma^n)\phi'(\gamma_0)$. Dividing by n and taking the limit, we get $\phi' = \phi'(\gamma_0).\phi$.

We now show that the action of Γ is semi-conjugate to a group of translations. If Γ is isomorphic to $\mathbf{Z}$, it acts freely and properly on the line so that it is indeed conjugate to the group of integral translations. Otherwise, $\phi(\Gamma)$ is dense in $\mathbf{R}$. Let x be any point in $\mathbf{R}$ and define

$$h(x) = \sup\{\phi(\gamma) \in \mathbf{R} \mid \gamma(0) \leq x\}.$$

Clearly, h is non decreasing and satisfies $h(\gamma(x)) = h(x) + \phi(\gamma)$ identically. The continuity of h is easy and follows from the density of the group $\phi(\Gamma)$: if h were not continuous, the interior of $\mathbf{R} \setminus h(\mathbf{R})$ would be a non empty open set invariant by all translations in $\phi(\Gamma)$.

The proof for groups acting on the circle follows easily: if Γ is a group acting freely on the circle, its inverse image in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ acts freely on the line. □

The following is an elementary corollary of the previous theorem.

PROPOSITION 6.11. *Let Γ be a torsion group (i.e. such that every element in Γ has finite order). Then any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has abelian image.*

Proof. We know the structure of elements of finite order of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to rotations of finite order. It follows that an element having a fixed point and of finite order in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is the identity. In other words, a torsion group acting faithfully on the circle acts freely. The result follows from 6.10. □

There is another very interesting example of a group which admits a left and right invariant total ordering: the group $\mathrm{PL}_+([0,1])$ of orientation preserving piecewise linear homeomorphisms of the interval $[0,1]$. Indeed, let γ, γ' be two distinct elements of $\mathrm{PL}_+([0,1])$ and consider the largest real number $x \in [0,1]$ such that γ and γ' coincide on the interval $[0,x]$. Then for $\epsilon > 0$ small enough, we have either $\gamma(t) < \gamma'(t)$ for $t \in]x, x+\epsilon]$ or $\gamma(t) > \gamma'(t)$ for $t \in]x, x+\epsilon]$. Say that $\gamma \prec \gamma'$ in the first case and $\gamma' \prec \gamma$ in the second case. This defines a total ordering on $\mathrm{PL}_+([0,1])$ and it is clearly left and right invariant. We can induce this ordering on countable subgroups of $\mathrm{PL}_+([0,1])$, for instance the subgroup of elements with rational slopes and apply the general construction that we described above. We get an action of this rational group on the line which is very different from the given action of $\mathrm{PL}_+([0,1])$ on $]0,1[$: the corresponding graphs don't cross the diagonal.

Remark that an affine bijection of the line $x \mapsto ax+b$ has at most one fixed point (if it is not the identity). Solodov proved that this property essentially characterizes groups of affine transformations.

THEOREM 6.12 (Solodov). *Let* Γ *be a non abelian subgroup of* $\mathrm{Homeo}_+(\mathbf{R})$ *such that every element (different from the identity) has at most one fixed point. Then* Γ *is isomorphic to a subgroup of the affine group* $\mathrm{Aff}_+(\mathbf{R})$ *of the real line, and the action of* Γ *on the line is semi-conjugate to the corresponding affine action.*

Solodov did not publish a proof but mentions his result in [62] and explained it to the author of these notes in 1991. Later T. Barbot needed this theorem for his study of Anosov flows and published a proof in [3]. More recently, N. Kovačević published an independent proof in [43]. See also the recent preprint [20] for a detailed proof.

Proof. Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbf{R})$ such that every element (different from the identity) has at most one fixed point. If no non trivial element has a fixed point, Hölder's Theorem 6.10 implies that Γ is abelian (and that the action is semi-conjugate to a group of translations). If there is a point x which is fixed by the full group Γ, then one can restrict the action to the two components of $\mathbf{R} \setminus \{x\}$ on which we can use Hölder's theorem again: this would imply that Γ is abelian.

We claim that Γ *contains an element* γ *with a repulsive fixed point* x*, i.e. such that* $\gamma(y) > y$ *for every* $y > x$ *and* $\gamma(y) < y$ *for every* $y < x$. Indeed choose some non trivial γ_0 in Γ fixing some x_0. If x_0 is not repulsive for γ_0 and for γ_0^{-1}, this means that x_0 is a parabolic fixed point, *i.e.* replacing γ_0 by its inverse, we have $\gamma_0(y) > y$ for all $y \neq x_0$. Conjugating γ_0 by some element which does not fix x_0, we get an element γ_1 fixing some x_1 and such that $\gamma_1(y) > y$ for $y \neq x_1$. Assume for instance $x_0 < x_1$ and consider the element $\gamma = \gamma_0\gamma_1^{-1}$. Obviously, one has $\gamma(x_0) < x_0$ and $\gamma(x_1) > x_1$ and since we know that γ has at most one fixed point, γ must have a repulsive fixed point between x_0 and x_1 as we claimed.

Now, we can try to mimic the proof of Hölder's theorem. Consider two elements γ and γ' of Γ. Write $\gamma \preceq \gamma'$ if there is some $x \in \mathbf{R}$ such that $\gamma(y) \leq \gamma'(y)$ for all $y > x$. Clearly, our assumptions imply that this defines a total ordering on Γ which is left and right-invariant. Denote by Γ^+ the subset of elements of $\Gamma \setminus \{id\}$ such that $id \preceq \gamma$.

The next claim is a weak form of the archimedean property. *Fix some γ_0 in Γ^+ with a repulsive fixed point x_0, and let γ be any other element of Γ^+. Then there exists some positive integer k such that $\gamma \preceq \gamma_0^k$.* Indeed, choose some real numbers x_-, x_+ such that $x_- < x_0 < x_+$. For k big enough, one has $\gamma_0^k(x_-) < \gamma(x_-)$ and $\gamma_0^k(x_+) > \gamma(x_+)$ since x_0 is repulsive. It follows that $\gamma^{-1}\gamma_0^k$ has a fixed point in the interval $[x_-, x_+]$ which is therefore the unique fixed point of $\gamma^{-1}\gamma_0^k$. Hence we have $\gamma_0^k(y) > \gamma^{-1}(y)$ for all $y > x_+$ and $\gamma \preceq \gamma_0^k$. This proves our last claim.

Again, we fix some γ_0 in Γ^+ with a repulsive fixed point x_0. For each $\gamma \in \Gamma^+$ we define $\Phi(\gamma) \in \mathbf{N}$ to be the smallest integer k such that $\gamma \preceq \gamma_0^k$. If $\gamma^{-1} \in \Gamma^+$, we let $\Phi(\gamma) = -\Phi(\gamma^{-1})$ and finally we define $\Phi(id) = 0$. This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$. Then we can copy from the proof of Hölder's theorem: Φ is a quasi-homomorphism and the limit $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a group homomorphism $\phi\colon \Gamma \to \mathbf{R}$.

It follows in particular that the first commutator group $[\Gamma, \Gamma]$ is contained in the kernel of ϕ. The final observation is that this kernel acts freely on the line. Otherwise, we saw that *Ker*(ϕ) would contain some element γ with a repulsive fixed point and we have already observed that this implies the existence of some integer k such that $\gamma_0 \preceq \gamma^k$ which in turn implies that $\phi(\gamma) \geq 1/k \neq 0$, a contradiction. Using Hölder's theorem, we conclude that $[\Gamma, \Gamma]$ is abelian.

We know the structure of free actions (of abelian groups) on the line: they are semi-conjugate to translation groups. More precisely, we know that there is a map $h\colon \mathbf{R} \to \mathbf{R}$ and an injective homomorphism $\psi\colon [\Gamma, \Gamma] \to \mathbf{R}$ which are such that for every $\gamma \in [\Gamma, \Gamma]$ and $x \in \mathbf{R}$, one has: $h(\gamma(x)) = h(x) + \psi(\gamma)$. If the image $\psi([\Gamma, \Gamma])$ is non discrete, this map h is unique up to post-composition by an affine map. So assume first that $\psi([\Gamma, \Gamma])$ is non discrete. Note that $[\Gamma, \Gamma]$ is a normal subgroup of Γ. It follows that for every γ in Γ, the map $h \circ \gamma$ coincides with h up to some affine map. This means precisely that h realizes a semi-conjugacy between Γ and some group of affine transformations of $\mathbf{R}$ and shows that Γ is indeed isomorphic to a subgroup of Aff$(\mathbf{R})$. To finish the proof, we still have to show that $\psi([\Gamma, \Gamma])$ cannot be discrete, *i.e.* isomorphic to $\mathbf{Z}$. In this case, inner conjugacies by an element $\gamma \in \Gamma$ have to preserve the generator 1 of $\mathbf{Z}$ (the unique generator which is bigger that the identity in our ordering). This means that $\mathbf{Z}$ ($\simeq [\Gamma, \Gamma]$) lies in the center of Γ. This is not possible since for every fixed point x of an element γ of Γ, its orbit under $\mathbf{Z}$ would consist of fixed points of γ. □

Hölder's theorem essentially characterizes translation groups as groups acting on the line with no fixed points. Solodov's theorem essentially characterizes groups of affine transformations as groups acting on the line with at most one fixed point. It is very tempting to try to prove a similar characterization of groups of projective transformations as groups acting on the circle with at most two fixed points... Unfortunately, this is not the case! N. Kovačević recently constructed a nice counter-example in [44].

THEOREM 6.13 (Kovačević). *There exists a finitely generated subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that every element different from the identity has at most two fixed points, such that all orbits are dense, and which is not conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$.

Nevertheless, there is a very important characterization of groups which are conjugate to subgroups of $\mathrm{PSL}(2,\mathbf{R})$. This characterization is due to Casson-Jungreis and Gabai [15, 24], following earlier work of Tukia. We would have liked to include a discussion and a proof of this result, but that would be too long and we have to limit ourselves to a statement! Consider a sequence γ_n of elements of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Let us say that γ_n has the *convergence property* if it contains a subsequence γ_{n_k} which satisfies one of the following two properties:

- γ_{n_k} is equicontinuous;
- there exist two points x, y on the circle such that γ_{n_k} (resp. $\gamma_{n_k}^{-1}$) converges to a constant map on each compact interval in $\mathbf{S}^1 \setminus \{x\}$ (resp. in $\mathbf{S}^1 \setminus \{y\}$).

A subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$ is called a *convergence group* if every sequence of elements of Γ has the convergence property.

THEOREM 6.14 (Casson-Jungreis, Gabai). *A subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *if and only if it is a convergence group.*

The reader should at least be able to prove the easy part of the theorem: subgroups of $\mathrm{PSL}(2,\mathbf{R})$ are convergence groups!

We revert now to groups acting on the circle. We state a general criterion which characterizes the bounded classes coming from some action.

THEOREM 6.15 ([25]). *Let* Γ *be a countable group and* c *a class in* $H_b^2(\Gamma, \mathbf{Z})$. *Then there exists a homomorphism* $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ *such that* $\phi^*(eu) = c$ *if and only if* c *can be represented by a cocycle which takes only the values* 0 *and* 1.

Proof. Of course, the necessary condition is clear from 6.3 and the main difficulty will be to construct some action from a cocycle taking two values. Let c be a 2-cocycle on the group Γ taking only the values 0 and 1. We saw that a central extension and a section lead to a 2-cocycle. The process can be reversed and we can construct a central extension $\widetilde{\Gamma}$ in the following way from a 2-cocycle c. As a set, $\widetilde{\Gamma}$ is the product $\mathbf{Z} \times \Gamma$ and we define a multiplication $\bullet$ by:

$$(n_1, \gamma_1) \bullet (n_2, \gamma_2) = (n_1 + n_2 + \bar{c}(\gamma_1, \gamma_2), \gamma_1\gamma_2)$$

where, as usual, $\bar{c}$ denotes the inhomogeneous cocycle associated to c. The fact that $\widetilde{\Gamma}$ is a group is a restatement of the fact that c is a cocycle. The projection $\widetilde{\Gamma} \to \Gamma$ is a group homomorphism.

Assume first that the cocycle c *is non degenerate*, *i.e.* that $\bar{c}(id, \gamma) = \bar{c}(\gamma, id) = 0$ for every γ in Γ (where id denotes the identity element in Γ). Then the identity element of $\widetilde{\Gamma}$ is $(0, id)$ and the map $n \in \mathbf{Z} \mapsto (n, id) \in \widetilde{\Gamma}$ is also a group homomorphism. Hence, we have a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1\,.$$

The fact that c takes non negative values means that the subset P of $\widetilde{\Gamma}$ consisting of elements of the form (n, γ) with $n \geq 0$ is a semi-group, *i.e.* is stable under the product $\bullet$. Moreover, since c takes the values 0 and 1, the inverse of (n, γ) is $(-n, \gamma^{-1})$ or $(-n-1, \gamma^{-1})$. It follows that every element of $\widetilde{\Gamma}$ belongs to P or to its inverse. In other words, if one defines $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if $\widetilde{\gamma}_2\widetilde{\gamma}_1^{\,-1} \in P$ we get a total pre-order on $\widetilde{\Gamma}$ which is left invariant. Denote by $\mathbf{t}$ the element $(1, id)$ in $\widetilde{\Gamma}$. Note that for every $\widetilde{\gamma}$ in $\widetilde{\Gamma}$ we have $\widetilde{\gamma} \preceq \mathbf{t}\widetilde{\gamma}$.

The end of the proof mimics 6.8: One constructs a map $v\colon \widetilde{\Gamma} \to \mathbf{R}$ such that $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if and only if $v(\widetilde{\gamma}_1) \leq v(\widetilde{\gamma}_2)$ and such that $v(\widetilde{\gamma}\mathbf{t}) = v(\widetilde{\gamma}) + 1$ for every $\widetilde{\gamma} \in \widetilde{\Gamma}$. We may even choose v in such a way that the action of $\widetilde{\Gamma}$ on itself by left translations defines an action on $v(\widetilde{\Gamma}) \subset \mathbf{R}$ which extends to its closure. Then we extend this action of $\widetilde{\Gamma}$ to $\mathbf{R}$ using affine maps in the connected components of the complement of this closure. Finally, since $\mathbf{t}$ acts on $\mathbf{R}$ by the translation by 1, we get an action of the quotient group Γ on the circle $\mathbf{R}/\mathbf{Z}$. This construction was carried out in such a way that it is clear that the bounded Euler class of this action is precisely the class of the cocycle c.

Finally, we have to deal with the case of degenerate cocycles c. Note that the fact that c is a cocycle can be expressed by the identity:

$$\bar{c}(\gamma_1, \gamma_2) + \bar{c}(\gamma_1\gamma_2, \gamma_3) = \bar{c}(\gamma_2, \gamma_3) + \bar{c}(\gamma_1, \gamma_2\gamma_3) .$$

It follows that there exists an integer $\nu = 0$ or 1 such that for every γ in Γ we have $\bar{c}(1, \gamma) = \bar{c}(\gamma, 1) = \nu$. The fact that c is degenerate means that $\nu = 1$. Then we can define $c' = 1 - c$. This is a new cocycle which is non degenerate and takes only the values 0 and 1. By the previous construction, we get an action of Γ on the circle corresponding to the bounded class of c'. Reversing the orientation of the circle, we get finally an action of Γ on the circle whose bounded Euler class is the class of c. □

6.6 Some examples

Recall that a group Γ is called *perfect* if every element is a product of commutators. It is *uniformly perfect* if there is an integer k such that every element is a product of at most k commutators. For such a uniformly perfect group, every quasi-homomorphism from Γ to $\mathbf{R}$ is bounded (since it is bounded on a single commutator) so that the canonical map from $H_b^2(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is injective. Moreover the map from $H_b^2(\Gamma, \mathbf{Z})$ to $H_b^2(\Gamma, \mathbf{R})$ is also injective since there is no homomorphism from Γ to $\mathbf{R}$. In such a situation, the usual Euler class in $H^2(\Gamma, \mathbf{Z})$ determines the bounded Euler class, and therefore most of the topological dynamics of a group action.

An example of such a group is $\mathrm{SL}(n, \mathbf{Z})$ which is uniformly perfect for $n \geq 3$ and which, moreover is such that $H^2(\mathrm{SL}(n, \mathbf{Z}), \mathbf{Z}) = 0$ (for $n \geq 3$) [52]. As a corollary, we get immediately that *for $n \geq 3$, any action of* $\mathrm{SL}(n, \mathbf{Z})$ *on the circle has a fixed point.* This will be strengthened later in 7.1. Some other matrix groups have this property: see for instance [5, 14].

Consider the case of the Thompson group G. We can show that every element in G is a product of two commutators (see [28]) and that $H^2(G, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. Using the Milnor-Wood inequality we can show that in $H^2(G, \mathbf{Z})$ only the elements $-1, 0, +1$ have a norm less than or equal to $1/2$. Hence we deduce that *any non-trivial action of the Thompson group G on the circle is semi-conjugate to the canonical action given by its embedding in* $\mathrm{PL}_+(\mathbf{S}^1)$ or to the reverse embedding obtained by conjugating by an orientation reversing homeomorphism of the circle (see [28] for more details).