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Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_g = \big\langle z, a_1, b_1, \ldots, a_g, b_g \mid \\ a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \big\rangle .$$

The central subgroup A generated by z turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of Γ_g by $\mathbf{Z}$, hence an Euler class in $H^2(\Gamma_g, \mathbf{Z})$. It is a fact that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic with $\mathbf{Z}$ and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \geq 1$ has a contractible universal cover and that the cohomology of Γ_g can therefore be identified with the cohomology of the compact oriented surface of genus g (see [11] for more details).

6.2 The Euler class of a group action on the circle

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1 .$$

The cohomology group $H^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to $\mathbf{Z}$ and a generator is the Euler class of this central extension [50].

Consider now a homomorphism ϕ from some group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. Then, we can pull back the previous extension by ϕ. In other words, we consider the set of $(\gamma, \widetilde{f}) \in \Gamma \times \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi(\gamma) = p(\widetilde{f})$. This is a group $\widetilde{\Gamma}$ equipped with a canonical projection onto Γ whose kernel is isomorphic to $\mathbf{Z}$, *i.e.* $\widetilde{\Gamma}$ is a central extension of Γ by $\mathbf{Z}$. In case ϕ is injective, $\widetilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under p, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of Γ is called *the Euler class of the homomorphism* ϕ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms ϕ_1 and ϕ_2 have the same Euler class in $H^2(\Gamma, \mathbf{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism ϕ lifts to a homomorphism $\widetilde{\phi} \colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi = p \circ \widetilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when Γ is free.

If Γ_g is the fundamental group of a closed oriented surface of genus $g \geq 1$, we know that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ so that the Euler class

$eu(\phi)$ in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each $1 \leq i \leq g$, choose lifts $\widetilde{a}_i$ and $\widetilde{b}_i$ of $\phi(a_i)$ and $\phi(b_i)$. Now compute the product of commutators $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\dots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number $eu(\phi)$.

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus $g > 1$ can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space $\mathcal{H}$ can be equipped with a metric of curvature -1 whose group of orientation preserving isometries is precisely $\mathrm{PSL}(2, \mathbf{R})$. Moreover, any complete simply connected riemannian surface of curvature -1 is isometric to $\mathcal{H}$. Hence there are embeddings ϕ of the fundamental group Γ_g of a closed oriented surface of genus $g > 1$ in $\mathrm{PSL}(2, \mathbf{R})$ such that the corresponding action of Γ_g on $\mathcal{H}$ is free, proper and cocompact. Since we know that $\mathrm{PSL}(2, \mathbf{R})$ is a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can compute the corresponding Euler number $eu(\phi)$. The result of the computation is $2g - 2$. Note that each element of $\phi(\Gamma_g)$ is hyperbolic since the action is free and cocompact so that the rotation number of every element of $\phi(\Gamma_g)$ is 0. So we are in a situation in which the topological invariant $eu(\phi)$ is not 0 but the rotation number invariants are trivial; a situation different from the case where $\Gamma = \mathbf{Z}$.

6.3 Bounded cohomology and the Milnor-Wood inequality

It was observed very early that the Euler class of a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ cannot be arbitrary. Milnor and Wood proved the following [51, 71].

THEOREM 6.1 (Milnor-Wood). *Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$ and $\phi\colon \Gamma_g \to \mathrm{Homeo}_+(\mathbf{S}^1)$ be any homomorphism. Then the Euler number satisfies $|eu(\phi)| \leq 2g - 2$.*

Proof. We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that $eu(\phi)$ is the translation number of the homeomorphism $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\dots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. We also know that the translation number function τ is a quasi-homomorphism, *i.e.* there is some inequality of the form $\left|\tau(\widetilde{f}_1\widetilde{f}_2) - \tau(\widetilde{f}_1) - \tau(\widetilde{f}_2)\right| \leq D$ for some D. We also know that $\tau\left(\widetilde{f}^{-1}\right) = -\tau(\widetilde{f})$. So, if we evaluate τ on this element, we get a bound of the form $|eu(\phi)| \leq (4g-1)D$. This is not quite the bound given in the