Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	47 (2001)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	GROUPS ACTING ON THE CIRCLE
Autor:	GHYS, Étienne
Kapitel:	6.2 The Euler class of a group action on the circle
DOI:	https://doi.org/10.5169/seals-65441

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

Download PDF: 09.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_{g} = \left\langle z, a_{1}, b_{1}, \dots, a_{g}, b_{g} \right| \\ a_{1}b_{1}a_{1}^{-1}b_{1}^{-1}\dots a_{g}b_{g}a_{g}^{-1}b_{g}^{-1} = z, \quad za_{i} = a_{i}z, \quad zb_{i} = b_{i}z \right\rangle.$$

The central subgroup A generated by z turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of Γ_g by Z, hence an Euler class in $H^2(\Gamma_g, \mathbb{Z})$. It is a fact that $H^2(\Gamma_g, \mathbb{Z})$ is isomorphic with Z and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \ge 1$ has a contractible universal cover and that the cohomology of Γ_g can therefore be identified with the cohomology of the compact oriented surface of genus g (see [11] for more details).

6.2 The Euler class of a group action on the circle

We have already met a central extension related to groups of homeomorphisms

 $0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\operatorname{Homeo}}_+(\mathbf{S}^1) \stackrel{p}{\longrightarrow} \operatorname{Homeo}_+(\mathbf{S}^1) \longrightarrow 1.$

The cohomology group $H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to \mathbf{Z} and a generator is the Euler class of this central extension [50].

Consider now a homomorphism ϕ from some group Γ to Homeo₊(S¹). Then, we can pull back the previous extension by ϕ . In other words, we consider the set of $(\gamma, \tilde{f}) \in \Gamma \times \operatorname{Homeo_+}(S^1)$ such that $\phi(\gamma) = p(\tilde{f})$. This is a group $\tilde{\Gamma}$ equipped with a canonical projection onto Γ whose kernel is isomorphic to \mathbb{Z} , *i.e.* $\tilde{\Gamma}$ is a central extension of Γ by \mathbb{Z} . In case ϕ is injective, $\tilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under p, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of Γ is called *the Euler class* of the homomorphism ϕ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbb{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms ϕ_1 and ϕ_2 have the same Euler class in $H^2(\Gamma, \mathbb{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism ϕ lifts to a homomorphism $\tilde{\phi}: \Gamma \to \operatorname{Homeo_+}(S^1)$ such that $\phi = p \circ \tilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when Γ is free.

If Γ_g is the fundamental group of a closed oriented surface of genus $g \ge 1$, we know that $H^2(\Gamma_g, \mathbb{Z})$ is isomorphic to \mathbb{Z} so that the Euler class

 $eu(\phi)$ in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each $1 \le i \le g$, choose lifts \tilde{a}_i and \tilde{b}_i of $\phi(a_i)$ and $\phi(b_i)$. Now compute the product of commutators $\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}\tilde{b}_1^{-1}\ldots\tilde{a}_g\tilde{b}_g\tilde{a}_g^{-1}\tilde{b}_g^{-1}$. Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number $eu(\phi)$.

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus g > 1 can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space \mathcal{H} can be equipped with a metric of curvature -1 whose group of orientation preserving isometries is precisely PSL(2, **R**). Moreover, any complete simply connected riemannian surface of curvature -1 is isometric to \mathcal{H} . Hence there are embeddings ϕ of the fundamental group Γ_g of a closed oriented surface of genus g > 1 in PSL(2, **R**) such that the corresponding action of Γ_g on \mathcal{H} is free, proper and cocompact. Since we know that PSL(2, **R**) is a subgroup of Homeo₊(**S**¹), we can compute the corresponding Euler number $eu(\phi)$. The result of the computation is 2g - 2. Note that each element of $\phi(\Gamma_g)$ is hyperbolic since the action is free and cocompact so that the rotation number of every element of $\phi(\Gamma_g)$ is 0. So we are in a situation in which the topological invariant $eu(\phi)$ is not 0 but the rotation number invariants are trivial; a situation different from the case where $\Gamma = \mathbf{Z}$.

6.3 BOUNDED COHOMOLOGY AND THE MILNOR-WOOD INEQUALITY

It was observed very early that the Euler class of a homomorphism $\phi: \Gamma \rightarrow \text{Homeo}_+(\mathbf{S}^1)$ cannot be arbitrary. Milnor and Wood proved the following [51, 71].

THEOREM 6.1 (Milnor-Wood). Let Γ_g be the fundamental group of a closed oriented surface of genus $g \ge 1$ and $\phi: \Gamma_g \to \text{Homeo}_+(\mathbf{S}^1)$ be any homomorphism. Then the Euler number satisfies $|eu(\phi)| \le 2g - 2$.

Proof. We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that $eu(\phi)$ is the translation number of the homeomorphism $\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \dots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1}$. We also know that the translation number function τ is a quasi-homomorphism, *i.e.* there is some inequality of the form $|\tau(\tilde{f}_1 \tilde{f}_2) - \tau(\tilde{f}_1) - \tau(\tilde{f}_2)| \leq D$ for some D. We also know that $\tau(\tilde{f}^{-1}) = -\tau(\tilde{f})$. So, if we evaluate τ on this element, we get a bound of the form $|eu(\phi)| \leq (4g-1)D$. This is not quite the bound given in the