

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 47 (2001)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GROUPS ACTING ON THE CIRCLE  
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**Kapitel:** 6.1 Group cohomology  
**DOI:** <https://doi.org/10.5169/seals-65441>

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of  $\Gamma$  since, once again,  $\Gamma$  commutes with  $\theta$ .

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of  $\Gamma$  on  $S^1$ . As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in  $[x, \theta(x)]$  is contractible, by definition. This means that any compact interval in  $S^{1'}$  is contractible and this implies (STRONG EXPANSIVITY).

*We have now proved that if (MINIMALITY) and (EXPANSIVITY) are both satisfied, then the group  $\Gamma$  must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group  $\Gamma$ . If a group  $\Gamma$  acts equicontinuously, then its closure in  $\text{Homeo}_+(S^1)$  is a compact group by Ascoli's theorem. We analyzed compact subgroups of  $\text{Homeo}_+(S^1)$  in 4.1: they turned out to be abelian and conjugate to groups of rotations.

*We have shown that if (MINIMALITY) holds then  $\Gamma$  is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let  $\Gamma$  be any subgroup of  $\text{Homeo}_+(S^1)$  and let us use the structure theorem 5.6–5.8. If  $\Gamma$  is minimal, we have already proved the theorem. If  $\Gamma$  has a finite orbit, there is a  $\Gamma$ -invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the  $\Gamma$ -action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that  $\Gamma$  contains a non abelian free subgroup unless the restriction of the action of  $\Gamma$  to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a  $\Gamma$ -invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

## 6. BOUNDED EULER CLASS

### 6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let  $\Gamma$  be any group. Let us consider the (semi)-simplicial set  $E\Gamma$  whose vertices are the elements of  $\Gamma$  and for which  $n$ -simplices are all  $(n+1)$ -tuples of elements of  $\Gamma$ . The  $i^{\text{th}}$  face of the simplex  $(\gamma_0, \dots, \gamma_k)$  is  $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$  where the term  $\gamma_i$  is omitted. Note that the set  $E\Gamma$  does not depend on the group structure of  $\Gamma$ .

As a matter of fact,  $E\Gamma$  is contractible since it is the full simplex over the set  $\Gamma$ . However, there is a simplicial free action of  $\Gamma$  on  $E\Gamma$  induced by left translations of  $\Gamma$  on itself. Hence one could think of the quotient  $B\Gamma$  of  $E\Gamma$  by this action as a space whose fundamental group is  $\Gamma$  and with vanishing higher homotopy groups. One would like to define the cohomology of the group  $\Gamma$  as the cohomology of this quotient space  $B\Gamma$ . We should be careful with  $B\Gamma$  since it has only one vertex (a group acts transitively on itself!).

However, guided by this idea, it is natural to define a  $k$ -cochain of  $\Gamma$  with values in some abelian group  $A$  as a map  $c: \Gamma^{k+1} \rightarrow A$  which is *homogeneous*, *i.e.* such that  $c(\gamma\gamma_0, \gamma\gamma_1, \dots, \gamma\gamma_k) = c(\gamma_0, \gamma_1, \dots, \gamma_k)$  identically. The set of these cochains is an abelian group denoted by  $C^k(\Gamma, A)$ . We have a natural coboundary  $d_k$  from  $C^k(\Gamma, A)$  to  $C^{k+1}(\Gamma, A)$  defined by

$$d_k c(\gamma_0, \dots, \gamma_{k+1}) = \sum_{i=0}^k (-1)^i c(\gamma_0, \dots, \widehat{\gamma_i}, \dots, \gamma_k).$$

Of course, we have  $d_{k+1} \circ d_k = 0$  and we define the *cohomology group*  $H^k(\Gamma, A)$  as being the quotient of cocycles (*i.e.* the kernel of  $d_k$ ) by coboundaries (*i.e.* the image of  $d_{k-1}$ ). If  $A$  is moreover a ring, then there is a natural cup product from  $H^k(\Gamma, A) \times H^l(\Gamma, A)$  to  $H^{k+l}(\Gamma, A)$ . We refer to [11] for an excellent account of this theory of group cohomology. Note that for any homomorphism  $\phi$  from a group  $\Gamma$  to another group  $\Gamma'$ , there is an induced homomorphism  $\phi^*: H^k(\Gamma', A) \rightarrow H^k(\Gamma, A)$ .

A homogeneous map  $c: \Gamma^{k+1} \rightarrow A$  can be written in a unique way in the form  $c(\gamma_0, \dots, \gamma_k) = \bar{c}(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \dots, \gamma_{k-1}^{-1}\gamma_k)$  for a unique function  $\bar{c}: \Gamma^k \rightarrow A$ . Conversely, given a map  $\bar{c}$  there is a unique homogeneous map  $c$  satisfying this relation. One says that  $\bar{c}$  is the inhomogeneous cochain associated to  $c$ . In other words, the space  $C^k(\Gamma, A)$  is canonically isomorphic to the  $A$ -module of all maps  $\Gamma^k \rightarrow A$ .

In degree 1, a cochain is a homogeneous map  $c: \Gamma^2 \rightarrow A$  and the corresponding inhomogeneous cochain is a map  $\bar{c}: \Gamma \rightarrow A$ . It is interesting to check that  $c$  is a cocycle if and only if  $\bar{c}$  is a homomorphism. Moreover 0-cochains are constant maps from  $\Gamma$  to  $A$  and their coboundary is therefore 0. It follows that *for any group  $\Gamma$ , the cohomology  $H^1(\Gamma, A)$  is identified with the set of homomorphisms from  $\Gamma$  to  $A$ .*

In degree 2, the interpretation is quite interesting. Consider a central extension of  $\Gamma$  by  $A$ :

$$0 \longrightarrow A \xrightarrow{i} \widetilde{\Gamma} \xrightarrow{p} \Gamma \longrightarrow 1.$$

This means that  $\tilde{\Gamma}$  contains a subgroup isomorphic to  $A$  contained in its center and that the quotient by this subgroup is isomorphic to  $\Gamma$ . Suppose that the projection  $p$  has a section  $s$  which is a homomorphism from  $\Gamma$  to  $\tilde{\Gamma}$  such that  $p \circ s = Id_{\Gamma}$ . Then it follows that  $\tilde{\Gamma}$  is isomorphic to the direct product  $\Gamma \times A$  by the homomorphism sending  $(\gamma, a)$  to  $s(\gamma)i(a)$ . Hence, in order to measure the non triviality of an extension we try to find the “obstruction” to finding a section  $s$ . This is done in the following way. Choose a set theoretical section  $s$  from  $\Gamma$  to  $\tilde{\Gamma}$ ; this is possible since  $p$  is onto. If  $\gamma_1$  and  $\gamma_2$  are two elements of  $\Gamma$ , consider  $\bar{c}(\gamma_1, \gamma_2) = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2)$ . This element projects on the identity element of  $\Gamma$  under  $p$  since  $p$  is a homomorphism; it is therefore an element of the image of  $i$  and can be identified with an element of  $A$ . This defines a map  $\bar{c}: \Gamma^2 \rightarrow A$ . Let  $c: \Gamma^3 \rightarrow A$  be the associated homogeneous cochain. One checks that  $c$  is a cocycle. Of course, the section  $s$  is not unique but another choice  $s'$  has the form  $s'(\gamma) = s(\gamma)i(u(\gamma))$  for some function  $u: \Gamma \rightarrow A$ . If one computes the cocycle  $c'$  associated to this new choice of a section  $s'$ , one finds that  $c' - c$  is the coboundary of the 1-cochain associated to the map  $u$ . It follows that the cohomology class of  $c$  in  $H^2(\Gamma, A)$  is well defined, *i.e.* does not depend on the choice of a section. This cohomology class is called the *Euler class of the extension* under consideration.

It is not difficult to check the following properties of the Euler class.

- 1) Two central extensions  $\tilde{\Gamma}_1$  and  $\tilde{\Gamma}_2$  of  $A$  by  $\Gamma$  are isomorphic by some isomorphism which is the identity on the central subgroup  $A$  and inducing the identity on the quotient  $\Gamma$  if and only if they have the same Euler class in  $H^2(\Gamma, A)$ .
- 2) Any class in  $H^2(\Gamma, A)$  corresponds to a central extension.

In short,  $H^2(\Gamma, A)$  parametrizes isomorphism classes of central extensions of  $A$  by  $\Gamma$ .

Before coming back to the dynamics of groups acting on the circle, let us consider a few simple examples.

If  $\Gamma = \mathbf{Z}$ , it is clear that every extension admits a section which is a homomorphism: it suffices to choose arbitrarily  $s(1)$  in  $p^{-1}(1)$  and to define  $s(n) = s(1)^n$  for  $n \in \mathbf{Z}$ . Hence, if  $\Gamma = \mathbf{Z}$  or more generally if  $\Gamma$  is a free group, we have  $H^2(\Gamma, A) = 0$ .

Let  $\Gamma_g$  be the fundamental group of a closed oriented surface of genus  $g \geq 1$ . It has a presentation of the form

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid a_1b_1a_1^{-1}b_1^{-1} \dots a_gb_ga_g^{-1}b_g^{-1} = 1 \rangle.$$

Now consider the group  $\tilde{\Gamma}_g$  defined by the presentation

$$\begin{aligned}\tilde{\Gamma}_g = \langle z, a_1, b_1, \dots, a_g, b_g \mid \\ a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \rangle.\end{aligned}$$

The central subgroup  $A$  generated by  $z$  turns out to be infinite cyclic so that  $\tilde{\Gamma}_g$  defines a central extension of  $\Gamma_g$  by  $\mathbf{Z}$ , hence an Euler class in  $H^2(\Gamma_g, \mathbf{Z})$ . It is a fact that  $H^2(\Gamma_g, \mathbf{Z})$  is isomorphic with  $\mathbf{Z}$  and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus  $g \geq 1$  has a contractible universal cover and that the cohomology of  $\Gamma_g$  can therefore be identified with the cohomology of the compact oriented surface of genus  $g$  (see [11] for more details).

## 6.2 THE EULER CLASS OF A GROUP ACTION ON THE CIRCLE

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\text{Homeo}_+(\mathbf{S}^1)} \xrightarrow{p} \text{Homeo}_+(\mathbf{S}^1) \longrightarrow 1.$$

The cohomology group  $H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$  has been computed. It is isomorphic to  $\mathbf{Z}$  and a generator is the Euler class of this central extension [50].

Consider now a homomorphism  $\phi$  from some group  $\Gamma$  to  $\text{Homeo}_+(\mathbf{S}^1)$ . Then, we can pull back the previous extension by  $\phi$ . In other words, we consider the set of  $(\gamma, \tilde{f}) \in \Gamma \times \widetilde{\text{Homeo}_+(\mathbf{S}^1)}$  such that  $\phi(\gamma) = p(\tilde{f})$ . This is a group  $\tilde{\Gamma}$  equipped with a canonical projection onto  $\Gamma$  whose kernel is isomorphic to  $\mathbf{Z}$ , *i.e.*  $\tilde{\Gamma}$  is a central extension of  $\Gamma$  by  $\mathbf{Z}$ . In case  $\phi$  is injective,  $\tilde{\Gamma}$  is just the pre-image of  $\phi(\Gamma)$  under  $p$ , which is the group of lifts of  $\phi(\Gamma)$ . The Euler class of this central extension of  $\Gamma$  is called *the Euler class of the homomorphism*  $\phi$  and denoted by  $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$ . It is obviously a dynamical invariant in the sense that two conjugate homomorphisms  $\phi_1$  and  $\phi_2$  have the same Euler class in  $H^2(\Gamma, \mathbf{Z})$ . Note that it follows from the definition that  $eu(\phi)$  is zero if and only if the homomorphism  $\phi$  lifts to a homomorphism  $\tilde{\phi}: \Gamma \rightarrow \widetilde{\text{Homeo}_+(\mathbf{S}^1)}$  such that  $\phi = p \circ \tilde{\phi}$ .

A few examples are in order. In the case of a single homeomorphism, *i.e.* when  $\Gamma = \mathbf{Z}$ , we saw that  $H^2(\mathbf{Z}, \mathbf{Z}) = 0$ . Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when  $\Gamma$  is free.

If  $\Gamma_g$  is the fundamental group of a closed oriented surface of genus  $g \geq 1$ , we know that  $H^2(\Gamma_g, \mathbf{Z})$  is isomorphic to  $\mathbf{Z}$  so that the Euler class