

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 47 (2001)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GROUPS ACTING ON THE CIRCLE  
**Autor:** GHYS, Étienne  
**Kapitel:** 5.1 Dynamics of a single homeomorphism  
**DOI:** <https://doi.org/10.5169/seals-65441>

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## 5. ROTATION NUMBERS

## 5.1 DYNAMICS OF A SINGLE HOMEOMORPHISM

The main invariant of homeomorphisms of the circle was introduced by H. Poincaré (it is still very interesting to read [59]).

Let us start with an element  $\tilde{f}$  of  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ , *i.e.* a homeomorphism of  $\mathbf{R}$  which commutes with integral translations. Observe that if two points  $x, x'$  in  $\mathbf{R}$  differ by at most 1, the same is true for their images by  $\tilde{f}$ . It follows that for any two points  $x, x'$ , the two numbers  $\tilde{f}(x) - x$  and  $\tilde{f}(x') - x'$  differ by at most 1. Let us define  $T(\tilde{f}) = \tilde{f}(0) = \tilde{f}(0) - 0$ . If  $\tilde{f}_1$  and  $\tilde{f}_2$  are two elements of  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ , we have  $T(\tilde{f}_1 \tilde{f}_2) = (\tilde{f}_1(\tilde{f}_2(0)) - \tilde{f}_2(0)) + (\tilde{f}_2(0) - 0)$  so that  $|T(\tilde{f}_1 \tilde{f}_2) - T(\tilde{f}_1) - T(\tilde{f}_2)|$  is bounded by 1. Let us formalize this notion:

**DEFINITION 5.1.** Let  $\Gamma$  be a group. A quasi-homomorphism from  $\Gamma$  to  $\mathbf{R}$  is a map  $F: \Gamma \rightarrow \mathbf{R}$  such that there is a constant  $D$  such that for every  $\gamma_1, \gamma_2$  in  $\Gamma$  we have  $|F(\gamma_1 \gamma_2) - F(\gamma_1) - F(\gamma_2)| \leq D$ .

The following is an easy exercise left to the reader.

**LEMMA 5.2.** *Let  $F: \mathbf{Z} \rightarrow \mathbf{R}$  be a quasi-homomorphism. Then, there exists a unique real number  $\tau$  such that the sequence  $F(n) - n\tau$  is bounded.*

As we shall see later, this lemma is far from being true if we replace the group  $\mathbf{Z}$  by a more general group  $\Gamma$ .

Let us restrict the quasi-homomorphism  $T$  to the group generated by a homeomorphism  $\tilde{f}$ , *i.e.* let us consider the sequence  $T(\tilde{f}^n)$ . According to the lemma, there is a unique number  $\tau(\tilde{f})$  such that  $T(\tilde{f}^n) - n\tau$  is bounded. This number  $\tau(\tilde{f})$  is by definition the *translation number* of  $\tilde{f}$ . It follows from the definition that if we compose  $\tilde{f}$  with an integral translation, the translation number increases by an integer. If we consider an element  $f$  in  $\text{Homeo}_+(\mathbf{S}^1)$ , the translations numbers of its lifts in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  differ by integers so that the element  $\rho(f) = \tau(\tilde{f}) \bmod \mathbf{Z} \in \mathbf{R}/\mathbf{Z}$  is well defined. This is called the *rotation number* of the homeomorphism  $f$ .

These definitions show that  $\tau$  is a quasi-homomorphism from  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  to  $\mathbf{R}$  and that it has been “normalized” so that it is a homomorphism on each one generator subgroup, *i.e.* we have  $\tau(\tilde{f}^n) = n\tau(\tilde{f})$  for every  $\tilde{f}$  and every integer  $n$ .

Of course, it is an easy matter to check that the translation number of the translation by  $\tau$  in  $\mathbf{R}$  is  $\tau$  and that the rotation number of the rotation  $x \in \mathbf{R}/\mathbf{Z} \mapsto x + \rho \in \mathbf{R}/\mathbf{Z}$  of “angle”  $\rho$  on the circle is indeed  $\rho$  as it should be !

The next proposition is easy but is a justification for introducing these numbers.

**PROPOSITION 5.3.** *The translation number and the rotation number are invariant under conjugation in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  and  $\text{Homeo}_+(\mathbf{S}^1)$  respectively.*

*Proof.* This follows formally from the fact that  $\tau$  is a quasi-homomorphism and is a homomorphism on one generator groups. Indeed,

$$\tau(\tilde{f}^n) = n\tau(\tilde{f})$$

and

$$\tau(\tilde{h}\tilde{f}^n\tilde{h}^{-1}) = \tau((\tilde{h}\tilde{f}\tilde{h}^{-1})^n) = n\tau(\tilde{h}\tilde{f}\tilde{h}^{-1})$$

differ by a bounded amount, independent of  $n$ , so that they must be equal. This shows that the translation number is a conjugacy invariant. The assertion concerning the rotation number follows immediately.  $\square$

Let us give some universal characterization of the translation number.

**PROPOSITION 5.4 ([4]).** *The translation number is the unique quasi-homomorphism  $\tau: \widetilde{\text{Homeo}}_+(\mathbf{S}^1) \rightarrow \mathbf{R}$  which is a homomorphism when restricted to one generator groups and which takes the value 1 on the translation by 1.*

*Proof.* An easy generalization of Lemma 5.2 shows that any quasi-homomorphism  $\mathbf{Z}^2 \rightarrow \mathbf{R}$  differs from a homomorphism  $\mathbf{Z}^2 \rightarrow \mathbf{R}$  by a bounded amount. This implies that if a quasi-homomorphism  $\mathbf{Z}^2 \rightarrow \mathbf{R}$  is a homomorphism when restricted to one generator groups, it is a homomorphism  $\mathbf{Z}^2 \rightarrow \mathbf{R}$  (note that a bounded homomorphism is necessarily trivial).

Let  $t$  be another quasi-homomorphism satisfying the conditions of the proposition. It follows from our first observation that  $t$  is a homomorphism when restricted to the (commutative) group generated by one element  $\tilde{f}$  and the integral translations. Consider now the difference  $r = \tau - t$ . Its value on an element  $\tilde{f}$  depends only on the projection of  $\tilde{f}$  in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  so that we get a quasi-homomorphism  $\bar{r}: \widetilde{\text{Homeo}}_+(\mathbf{S}^1) \rightarrow \mathbf{R}$  which is a homomorphism on one generator groups. We claim that  $\bar{r}$  must be trivial. This will follow from a property of  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  that we shall prove later (see 5.11): any homeomorphism  $f$  in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  can be written as a

commutator  $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$ . (In fact we only prove in 5.11 that any homeomorphism is a product of *two* commutators but this is enough for the proof which follows.) Assuming this result, we see that any quasi-homomorphism from  $\text{Homeo}_+(\mathbf{S}^1)$  has to be bounded. Indeed, up to a bounded amount, the value of the quasi-homomorphism  $\bar{r}$  on  $f = [f_1, f_2]$  is equal to the sum of its values on  $f_1, f_2, f_1^{-1}, f_2^{-1}$  which is bounded (since  $\bar{r}(f_1) + \bar{r}(f_1^{-1})$  is bounded). Now, a bounded quasi-homomorphism which is a homomorphism on one generator groups is trivial so that  $\bar{r}$  is zero.  $\square$

We mention a very interesting problem coming from [37]:

**PROBLEM 5.5 (Jenkins-Neumann).** *Let  $\mathcal{R} \subset (\mathbf{R}/\mathbf{Z})^3$  be the set of triples  $(\rho_1, \rho_2, \rho_3)$  such that there exist three elements  $f_1, f_2, f_3$  of  $\text{Homeo}_+(\mathbf{S}^1)$  such that  $f_1 f_2 f_3 = \text{Id}$  and whose rotation numbers are  $(\rho_1, \rho_2, \rho_3)$ . Can one describe this set  $\mathcal{R}$  explicitly?*

In [37], the authors show that  $\mathcal{R}$  has a fractal structure. First, they explicitly describe the set  $\mathcal{R}_0 \subset (\mathbf{R}/\mathbf{Z})^3$  of triples  $(\rho_1, \rho_2, \rho_3)$  such that there exist three elements  $f_1, f_2, f_3$  of some  $\text{PSL}_k(2, \mathbf{R})$  such that  $f_1 f_2 f_3 = \text{Id}$  and whose rotation numbers are  $(\rho_1, \rho_2, \rho_3)$ . Of course,  $\mathcal{R}_0 \subset \mathcal{R}$  and they conjecture that these two sets are equal. As a motivation for their conjecture, they find an explicit set  $\mathcal{R}_1$  such that  $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_1$  and such that  $\mathcal{R}_1 - \mathcal{R}_0$  is “small”: the Lebesgue measure of  $\mathcal{R}_1 - \mathcal{R}_0$  is indeed  $0.0010547\dots$  and the Lebesgue measure of  $\mathcal{R}_0$  is  $25/8 + 3\zeta(2) + 3\zeta(3) - 6\zeta(2)\zeta(3)/\zeta(5) \simeq 0.224649208402\dots$  (where  $\zeta$  is the Riemann  $\zeta$ -function). As Jenkins and Neumann write, their conjecture is therefore 99.9 % proved !

We shall show that the “number”  $\rho(f)$  contains a lot of information on the topological dynamics of  $f$ . Let us begin by explaining the main possibilities for the dynamics of an arbitrary group of homeomorphisms.

**PROPOSITION 5.6.** *Let  $\Gamma$  be any subgroup of  $\text{Homeo}_+(\mathbf{S}^1)$ . Then there are three mutually exclusive possibilities.*

- 1) *There is a finite orbit.*
- 2) *All orbits are dense.*
- 3) *There is a compact  $\Gamma$ -invariant subset  $K \subset \mathbf{S}^1$  which is infinite and different from  $\mathbf{S}^1$  and such that the orbits of points in  $K$  are dense in  $K$ . This set  $K$  is unique, contained in the closure of any orbit and is homeomorphic to a Cantor set.*

*Proof.* Let us consider the collection of compact sets in  $S^1$  which are non empty and  $\Gamma$ -invariant, ordered by inclusion. By Zorn's lemma, there is a minimal set in this collection. Choose such a *minimal set*  $K$ . Note that the closure of the orbit of any point in  $K$  is a closed non empty  $\Gamma$ -invariant set contained in  $K$  so that it must coincide with  $K$  by minimality: the orbit of a point in  $K$  is dense in  $K$ . Observe now that the topological boundary  $\partial K = K - \text{interior}(K)$  and the set  $K'$  of accumulation points of  $K$  are closed and  $\Gamma$ -invariant. Hence, we have the following possibilities.

- 1)  $K'$  is empty. In this case,  $K$  is finite and we found a finite orbit.
- 2)  $\partial K$  is empty, so that  $K$  is the full circle. In this case, all orbits are dense.
- 3)  $K' = K$  and  $\partial K = K$ , so that  $K$  is a compact perfect set in the circle with empty interior: this is one definition of a Cantor set.

In order to prove the uniqueness of  $K$  in the last case, we show that  $K$  is contained in the closure of any orbit. The complement of  $K$  in the circle is a disjoint union of a countable family of open intervals. Let  $x$  be a point in the complement of  $K$ , lying in some interval  $I$  and let  $a$  be the origin of  $I$  (note that  $I$  is oriented). Finally, let  $y$  be any point in  $K$ . Since we know that the orbit of any point of  $K$  is dense in  $K$  and that  $K$  has no isolated point, there is a sequence of elements  $\gamma_n$  such that  $\gamma_n(a)$  consists of distinct points and converges to  $y$ . The intervals  $\gamma_n(I)$  are therefore disjoint so that the distance between  $\gamma_n(a)$  and  $\gamma_n(x)$  converges to zero. It follows that  $\gamma_n(x)$  converges to  $y$ . This proves that  $K$  is contained in the closure of every orbit and the uniqueness of the minimal set  $K$  follows immediately.  $\square$

Case 3 looks strange at first sight: it is called the *exceptional minimal set* case for this reason. We reduce it to case 2, using the notion of *semi-conjugacy*. Consider a map  $\tilde{h}$  from  $\mathbf{R}$  to  $\mathbf{R}$  which is continuous, increasing (if  $x \leq y$  then  $\tilde{h}(x) \leq \tilde{h}(y)$ ) and which commutes with integral translations. We stress the fact that  $\tilde{h}$  might be non injective: typically it might be constant on some intervals. Such a map defines a map  $h$  from the circle to itself. We call such a map an *increasing continuous map of degree 1 from the circle to itself*.

**DEFINITION 5.7.** Let  $\Gamma$  be a group and  $\phi_1, \phi_2$  be two homomorphisms from  $\Gamma$  to  $\text{Homeo}_+(S^1)$ . We say that  $\phi_1$  is *semi-conjugate* to  $\phi_2$  if there is an increasing continuous map  $h$  of degree 1 from the circle to itself such that for every  $\gamma$  in  $\Gamma$ , we have  $\phi_2(\gamma)h = h\phi_1(\gamma)$ .

Observe that this notion is not symmetric:  $\phi_2$  is not necessarily semi-conjugate to  $\phi_1$ .

**PROPOSITION 5.8.** *Let  $\Gamma$  be a group and  $\phi$  be a homomorphism from  $\Gamma$  to  $\text{Homeo}_+(\mathbf{S}^1)$  such that  $\phi(\Gamma)$  has an exceptional minimal set  $K$ . Then there is a homomorphism  $\bar{\phi}$  from  $\Gamma$  to  $\text{Homeo}_+(\mathbf{S}^1)$  such that  $\phi$  is semi-conjugate to  $\bar{\phi}$  and  $\bar{\phi}(\Gamma)$  has dense orbits on the circle.*

*Proof.* The complement of  $K$  in the circle is a countable union of open intervals. For each of these intervals, collapse its closure to a point. The resulting quotient space is homeomorphic to a circle. In other words, there is an increasing continuous map  $h$  of degree 1 from the circle to itself such that  $h(K) = \mathbf{S}^1$  and such that the fibers  $h^{-1}(x)$  are either points or the closed intervals which are the closures of the connected components of the complement of  $K$ . Since  $\phi(\Gamma)$  acts on the circle and preserves  $K$ , it also acts on the “collapsed” circle so that we can define another homomorphism  $\bar{\phi}$  which satisfies the conditions of the proposition (we know that orbits of points in  $K$  are dense in  $K$ ).  $\square$

The main object of these notes is to discuss the dynamics of “big groups”  $\Gamma$  acting on the circle. However, we first restrict ourselves to the “easy” case where  $\Gamma$  is generated by one element so that we really study the dynamics of one homeomorphism of the circle. Of course, we allow ourselves to say that a homeomorphism  $f_1$  is semi-conjugate to  $f_2$  if the corresponding homomorphisms from  $\mathbf{Z}$  to  $\text{Homeo}_+(\mathbf{S}^1)$  are semi-conjugate. The following result shows that the rotation number of a homeomorphism contains a lot of information on the dynamics.

**THEOREM 5.9 (Poincaré).** *Let  $f$  be an element of  $\text{Homeo}_+(\mathbf{S}^1)$ . Then  $f$  has a periodic orbit if and only if the rotation number  $\rho(f)$  is rational, i.e. belongs to  $\mathbf{Q}/\mathbf{Z}$ . If the rotation number  $\rho(f)$  is irrational, then  $f$  is semi-conjugate to the rotation on the circle of angle  $\rho(f) \in \mathbf{R}/\mathbf{Z}$ . This semi-conjugacy is actually a conjugacy if the orbits of  $f$  are dense.*

*Proof.* Choose a lift  $\tilde{f}$  of  $f$  in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ . We know that the numbers  $\tilde{f}^n(x) - n\tau(\tilde{f})$  are uniformly bounded (independently of  $n \in \mathbf{Z}$ ). Define  $\tilde{h}(x) = \sup_n (\tilde{f}^n(x) - n\tau(\tilde{f}))$ . The following properties of  $\tilde{h}$  are obvious:

- 1)  $\tilde{h}$  is increasing (it is left continuous but not necessarily continuous).
- 2)  $\tilde{h}(x+1) = \tilde{h}(x) + 1$ .
- 3)  $\tilde{h}(\tilde{f}(x)) = \tilde{h}(x) + \tau(\tilde{f})$ .

If  $\tilde{h}$  were continuous, that would lead to a semi-conjugacy between  $f$  and the rotation by an angle  $\tau(\tilde{f}) \bmod \mathbf{Z} = \rho(f) \in \mathbf{R}/\mathbf{Z}$ . The structure of an increasing function like  $\tilde{h}$  from  $\mathbf{R}$  to  $\mathbf{R}$  is not difficult to analyze. First, the fibers  $\tilde{h}^{-1}(x)$  are either points or intervals. There is at most a countable number of these intervals: call the union of the interior of these intervals the plateau set  $Plat(\tilde{h})$  of  $\tilde{h}$ ; it is empty if and only if  $\tilde{h}$  is injective. Second, the image  $\tilde{h}(\mathbf{R})$  is the complement of the union of at most countably many disjoint intervals: call the union of the interior of these intervals the jump set  $Jump(\tilde{h})$  of  $\tilde{h}$ ; it is empty if and only if  $\tilde{h}$  is continuous and onto. In our situation it is clear from 2 that  $Plat(\tilde{h})$  and  $Jump(\tilde{h})$  are open sets which are invariant under integral translations, so that they define open sets in the circle. Moreover, property 3 shows that  $Plat(\tilde{h})$  is invariant under  $\tilde{f}$  and  $Jump(\tilde{h})$  is invariant under the translation by  $\tau(\tilde{f})$  so that the corresponding open sets in the circle are invariant under  $f$  and the rotation by an angle  $\rho(f)$  respectively.

We can now prove the theorem. Assume first that  $\tau(\tilde{f})$  is irrational so that all the orbits of the rotation of angle  $\rho(f)$  are dense. It follows that  $Jump(\tilde{h})$  has to be empty so that  $\tilde{h}$  defines a semi-conjugacy between  $f$  and the rotation by the angle  $\rho(f)$ . If the orbits of  $f$  are dense, then  $Plat(\tilde{h})$  has to be empty and  $\tilde{h}$  defines an actual conjugacy between  $f$  and the rotation. Also, since a semi-conjugacy maps finite orbit to finite orbit,  $f$  cannot have any periodic orbits in the case  $\rho(f)$  irrational.

Assume that  $\tau(\tilde{f})$  is a rational number of the form  $p/q$ . Then we know that the element  $\tilde{l}$  defined by  $\tilde{l}(x) = \tilde{f}^q - p$  has a vanishing translation number so that the orbit of any point  $x$  in  $\mathbf{R}$  under  $\tilde{l}$  is bounded. The upper bound of any orbit is a fixed point of  $\tilde{l}$ . Since  $\tilde{l}$  projects to  $f^q$  in  $Homeo_+(\mathbf{S}^1)$ , we have found a fixed point for  $f^q$ , hence a periodic orbit for  $f$ . This establishes the theorem. Note that in this last case, we showed something more: if  $\rho(f) = p/q \bmod \mathbf{Z}$ , then there is a periodic orbit whose “cyclic ordering” is the same as a rotation of angle  $p/q$ . This means that there is a homeomorphism  $h$  in  $Homeo_+(\mathbf{S}^1)$  whose *restriction* to the periodic orbit is a conjugacy between  $f$  and the rotation of angle  $p/q$ .  $\square$

We can now describe the dynamics of a homeomorphism  $f$  in  $Homeo_+(\mathbf{S}^1)$  quite precisely.

Suppose first that  $\rho(f)$  is *irrational*: we have two possibilities.

- 1) If all orbits are dense, then  $f$  is conjugate to the rotation of angle  $\rho(f)$ .
- 2) If there is an exceptional minimal set  $K \subset \mathbf{S}^1$  then  $f$  is semi-conjugate to the rotation of angle  $\rho(f)$ . The connected components of the complement of  $K$  are wandering intervals, *i.e.* disjoint from all their iterates.

It is not difficult to construct examples of the second type. Start with an irrational rotation of angle  $\rho$  on the circle and choose a (dense) orbit  $\mathcal{O} \subset \mathbf{S}^1$ . Then “blow up” each point in  $\mathcal{O}$  to replace it by an interval. In other words, consider a continuous increasing map  $h$  of degree 1 such that  $h^{-1}(x)$  is an interval if  $x$  is in  $\mathcal{O}$  and a point otherwise. The complement of the interior of these intervals is a Cantor set  $K$ . Then we construct a homeomorphism  $f$  of the circle which preserves  $K$ . On  $K$  the homeomorphism  $f$  is uniquely defined by the fact that  $h$  is a semi-conjugacy with the rotation. On the intervals of the complement of  $K$ , there is still some freedom in the construction: we choose any homeomorphism  $f$  which sends the interval  $h^{-1}(x)$  to the interval  $h^{-1}(x + \rho)$  for  $x$  in the orbit  $\mathcal{O}$ . The problem with this construction is that it is not clear whether or not we might do it in such a way that the corresponding homeomorphism  $f$  is smooth. Poincaré thought that there could exist an example of type 2 for which  $f$  is a real analytic diffeomorphism [59]: he was wrong, as shown later by Denjoy! Again, we refrain from discussing this point here since we decided to restrict these notes to topological problems.

Suppose now that  $\rho(f)$  is *rational* so that  $f$  has a periodic point. Replacing  $f$  by one of its powers  $f^q$ , we study the case where  $f$  has a fixed point. To understand the dynamics of  $f$ , we have first to describe the set  $Fix(f)$  of fixed points which can be an arbitrary compact set in the circle (so that it could be rather complicated). Then,  $f$  induces a homeomorphism of each connected component of the complement of  $Fix(f)$ . On each component  $f$  can move points “to the right” or “to the left” and this information is the only dynamical information: it is easy to show that up to orientation preserving conjugacy, there are two kinds of fixed point free homeomorphisms of an open interval, those going to the right and to the left respectively.

Summing up, we have a complete description of conjugacy classes of homeomorphisms of the circle. To give a complete list of invariants is possible but not very pleasant: for instance in the case of vanishing rotation number, we should describe a compact set up to homeomorphism and labels “left” or “right” on each component of the complement.

As a corollary, we get a description of those elements of  $\text{Homeo}_+(\mathbf{S}^1)$  which have the form  $\phi^t$  for some topological flow  $\phi^t$  on the circle. This follows immediately from our description of homeomorphisms and the description that we gave earlier of topological flows.

**PROPOSITION 5.10.** *An element  $f$  of  $\text{Homeo}_+(\mathbf{S}^1)$  can be included in a topological flow if and only if  $\rho(f) = 0$  or  $f$  is conjugate to a rotation.*

Note that it is possible to find elements  $f$  which are not included in flows arbitrarily close to the identity.

We can now prove an important fact that we have already used in the proof of the simplicity of  $\text{Homeo}_+(\mathbf{S}^1)$ .

**PROPOSITION 5.11.** *Every element of  $\text{Homeo}_+(\mathbf{S}^1)$  can be written as a product of two commutators.*

*Proof.* Consider a topological flow on the closed interval, *i.e.* a continuous homomorphism  $t \in \mathbf{R} \mapsto \phi^t \in \text{Homeo}_+([0, 1])$ . Assume that for  $t > 0$  the homeomorphism  $\phi^t$  satisfies  $\phi^t(x) > x$  for  $x \in ]0, 1[$ . By the previous discussion, all homeomorphisms  $\phi^t$  with  $t > 0$  are conjugate in  $\text{Homeo}_+([0, 1])$ . In particular, there is a homeomorphism  $l$  in  $\text{Homeo}_+([0, 1])$  such that  $l\phi^2l^{-1} = \phi^1$ . It follows that  $\phi^1 = \phi^2(\phi^1)^{-1} = \phi^2l(\phi^2)^{-1}l^{-1}$ . This shows that  $\phi^1$  is the commutator of  $\phi^2$  and  $l$ . Since we know that every homeomorphism of  $[0, 1]$  which fixes only 0 and 1 is conjugate to  $\phi^1$  or its inverse, it follows that any such homeomorphism can be written as a commutator.

We described the dynamics of homeomorphisms with rotation number  $0 \in \mathbf{R}/\mathbf{Z}$ : in each connected component of the complement of their non empty fixed point set, they are described by a homeomorphism of the closed interval with no fixed point in the interior. Our discussion therefore implies that every element of  $\text{Homeo}_+(\mathbf{S}^1)$  with rotation number 0 can be written as a commutator.

Consider finally an element  $f$  of  $\text{Homeo}_+(\mathbf{S}^1)$ . Clearly, one can choose a rotation  $r_\theta$  such that  $fr_\theta$  has a fixed point. In order to prove the proposition, it is therefore enough to show that any rotation can be written as a commutator. We show that this is indeed the case in  $\text{PSL}(2, \mathbf{R})$  using some hyperbolic geometry (of course, we could also prove the same thing by direct calculations).

Let  $a, b, c, d$  be four points in the Poincaré disc whose hyperbolic distances satisfy  $\text{dist}(a, b) = \text{dist}(c, d)$  and  $\text{dist}(a, d) = \text{dist}(b, c)$ . Let  $A$  (*resp.*  $B$ ) be the orientation preserving isometry of the Poincaré disc such that  $A(a) = b$ ,  $A(d) = c$  (*resp.*  $B(a) = d$ ,  $B(b) = c$ ). The commutator  $ABA^{-1}B^{-1}$  fixes the point  $c$ : it is therefore a hyperbolic rotation centered at the point  $c$ . Figure 5 shows that the angle of this rotation is equal to  $2\pi$  minus the sum of the angles of the quadrangle  $R = abcd$  which is equal to the area of this quadrangle and can take any value between 0 and  $2\pi$  since it is built out of two hyperbolic triangles. Hence we may realize any rotation as a single commutator.  $\square$

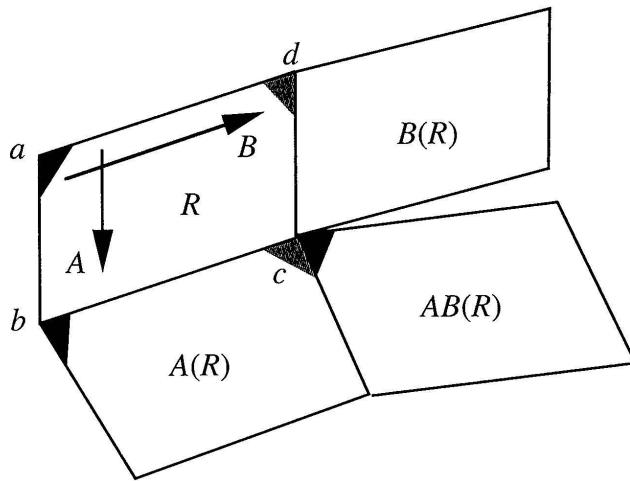


FIGURE 5

It turns out that Proposition 5.11 can be improved: every element of  $\text{Homeo}_+(\mathbf{S}^1)$  can be written as a single commutator. This is a special case of a result of [17] that we shall mention later in 6.2.

To conclude, we give some examples. Consider an element  $A$  of  $\text{PSL}(2, \mathbf{R})$  as a homeomorphism of the circle. The topological dynamics of  $A$  are easy to describe. Note that since  $A$  is a  $2 \times 2$  matrix up to sign, the absolute value of the trace of  $A$  is well defined. If  $|\text{tr}(A)| > 2$ , then  $A$  is called hyperbolic and has two fixed points on the circle. In this case, the rotation number of  $A$  is of course 0. If  $|\text{tr}(A)| = 2$ , then  $A$  is called parabolic and has only one fixed point; its rotation number is again 0. Finally, if  $|\text{tr}(A)| < 2$ , then  $A$  is called elliptic and is conjugate to (the equivalence class of) a rotation matrix by some angle  $2\pi\theta$  where  $\theta \in \mathbf{R}/\mathbf{Z}$  is such that  $2\cos(\theta) = |\text{tr}(A)|$ . In this case, the rotation number is therefore  $\cos^{-1}(\text{tr}(A)/2)/2\pi$ .

Let us consider a finitely generated fuchsian group  $\Gamma \subset \text{PSL}(2, \mathbf{R})$ . Since it is a discrete subgroup, any elliptic element in  $\Gamma$  must be of finite order. Assume that  $\Gamma$  is torsion free. (A theorem of Selberg guarantees that any finitely generated subgroup of a matrix group contains a finite index torsion free subgroup.) Then any element of  $\Gamma$  has rotation number equal to 0. However, there are many fuchsian groups exhibiting very rich dynamics, even with dense orbits. These examples show that *the data of all individual rotation numbers of the elements of a group acting on the circle is far from sufficient to describe the dynamics of the group*. In other words, Theorem 5.9 cannot be generalized so easily to groups more complicated than  $\mathbf{Z}$ . In the next section, we shall define a more subtle invariant suitable for bigger group actions, like fuchsian groups.

Observe that our computations in  $\text{PSL}(2, \mathbf{R})$  show that the rotation number

is a continuous function on  $\text{PSL}(2, \mathbf{R})$  but definitely not a smooth function. On the group  $\text{Homeo}_+(\mathbf{S}^1)$ , we have the following behaviour:

**PROPOSITION 5.12.** *The map  $\rho: \text{Homeo}_+(\mathbf{S}^1) \rightarrow \mathbf{R}/\mathbf{Z}$  is continuous and the pre-image of  $\mathbf{Q}/\mathbf{Z}$  contains an open and dense set.*

*Proof.* The continuity follows immediately from the definitions. Indeed, the continuous function  $\tilde{f} \mapsto \tilde{f}^n(0)/n$  on  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  differs at most by  $1/n$  from  $\tau(\tilde{f})$ . This implies the continuity of the translation number.

Suppose that  $\tilde{f}$  in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  is such that  $\tilde{f}(x) - x$  achieves both positive and negative values. Then  $\tilde{f}$  has at least a fixed point and  $\tau(\tilde{f}) = 0$ . Since this condition is obviously open in the uniform topology, we have found an open set on which the translation number takes the value 0. In the same manner, we construct open sets on which  $\tau$  takes the value  $p/q$ : the set of those  $\tilde{f}$  for which  $\tilde{f}^q(x) - x - p$  takes both positive and negative values.

We leave to the reader the (easy) proof that the set of  $\tilde{f}$  for which  $\tau(\tilde{f})$  is rational is dense in  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ .  $\square$

The local structure of the map  $\rho$  is quite interesting as was shown by a very nice example due to Arnold [1]. Consider the 2-parameter family of elements of  $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  given by

$$\tilde{f}_{\alpha, \epsilon}(x) = x + \alpha + \epsilon \sin(2\pi x).$$

Here  $\alpha$  is a real number and  $\epsilon$  is a real number which is small enough to guarantee that  $\tilde{f}_{\alpha, \epsilon}$  is a homeomorphism ( $|\epsilon| < 1/2\pi$  is enough). We should think of these  $\tilde{f}_{\alpha, \epsilon}$  as a small deformation of the translation by  $\alpha$  depending of the small parameter  $\epsilon$ . Let us look at the behaviour of  $\tau(\tilde{f}_{\alpha, \epsilon})$  as a function of  $\alpha$  and  $\epsilon$ . Of course, we have  $\tau(\tilde{f}_{\alpha, 0}) = \alpha$ . We can check rather easily the following facts. For each  $\epsilon$ , the function  $\alpha \mapsto \tau(\tilde{f}_{\alpha, \epsilon})$  is continuous and increasing but is not strictly increasing for  $\epsilon \neq 0$ . The plateau set of this function is the complement of a Cantor set on which  $\tau$  takes irrational values. The interior of the set of  $(\alpha, \epsilon)$  on which  $\tau$  takes the rational value  $p/q$  is an “Arnold tongue” which touches the axis  $\epsilon = 0$  at the point  $(p/q, 0)$ . The bigger the denominator  $q$ , the thinner the corresponding tongue.

Another interesting feature of this picture is that the Lebesgue measure of the set of  $(\alpha, \epsilon)$  for which  $\tau$  is irrational is not 0. Hence, the translation number takes rational values on an open dense set but takes irrational values on a set of positive Lebesgue measure.

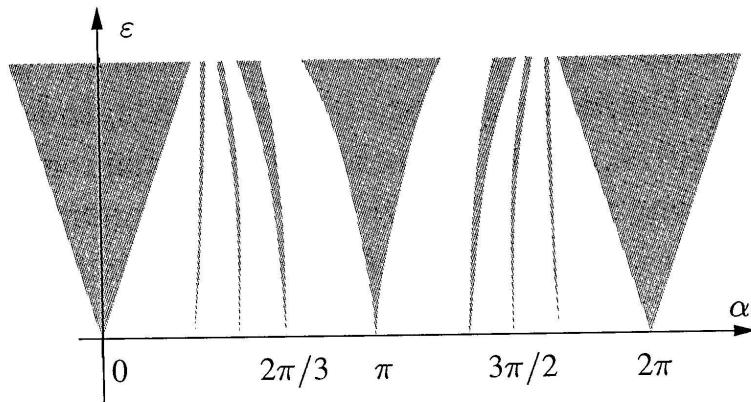


FIGURE 6

As an additional example, consider the case of piecewise linear homeomorphisms of the circle. Since the group  $\text{PL}_+(\mathbf{S}^1)$  contains all rotations, it is clear that the rotation number of such a homeomorphism can be arbitrary. However, it is shown in [28] that the rotation number of any element of the Thompson group  $G$  is rational and that any rational number is achieved. The proof is very indirect and there is a need for a better proof. We could formulate the problem in the following way.

**PROBLEM 5.13.** *Consider a rational piecewise linear homeomorphism  $f$  of the circle, i.e. such that all its slopes are rational and such that all “break-points” are rational. Is it true that the rotation number of  $f$  is rational?*

We can in fact generalize Thompson's group quite a lot in the following way. Let  $\Lambda \subset \mathbf{R}_*^+$  be a subgroup of the multiplicative group of positive real numbers and let  $W \subset \mathbf{R}$  be an additive subgroup invariant under multiplication by  $\Lambda$ . Then we can consider the subgroup  $\tilde{G}_{\Lambda, W}$  of  $\widetilde{\text{PL}}_+(\mathbf{S}^1)$  consisting of those elements with slopes in  $\Lambda$  and break-points in  $W$  (for instance, Thompson group is the case when  $\Lambda$  consists of powers of 2 and  $W$  of dyadic rationals). These groups are quite interesting especially when  $\Lambda$  is finitely generated (see [8, 9, 63]). It would be very useful to understand the nature of translation numbers of elements of  $\tilde{G}_{\Lambda, W}$  for specific  $\Lambda$  and  $W$ .

In [34], one can find (among other things !) a very interesting analysis of the rotation numbers of an explicit 1-parameter family of piecewise linear homeomorphisms of the circle.

## 5.2 TITS' ALTERNATIVE

Recall that J. Tits proved a remarkable alternative for finitely generated