5. Rotation numbers

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5. ROTATION NUMBERS

5.1 DYNAMICS OF A SINGLE HOMEOMORPHISM

The main invariant of homeomorphisms of the circle was introduced by H. Poincaré (it is still very interesting to read [59]).

Let us start with an element $\widetilde{f}$ of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$, *i.e.* a homeomorphism of $\mathbf{R}$ which commutes with integral translations. Observe that if two points x, x' in $\mathbf{R}$ differ by at most 1, the same is true for their images by $\widetilde{f}$. It follows that for any two points x, x', the two numbers $\widetilde{f}(x) - x$ and $\widetilde{f}(x') - x'$ differ by at most 1. Let us define $T(\widetilde{f}) = \widetilde{f}(0) = \widetilde{f}(0) - 0$. If $\widetilde{f}_1$ and $\widetilde{f}_2$ are two elements of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$, we have $T(\widetilde{f}_1\widetilde{f}_2) = (\widetilde{f}_1(\widetilde{f}_2(0)) - \widetilde{f}_2(0)) + (\widetilde{f}_2(0) - 0)$ so that $\left|T(\widetilde{f}_1\widetilde{f}_2) - T(\widetilde{f}_1) - T(\widetilde{f}_2)\right|$ is bounded by 1. Let us formalize this notion:

DEFINITION 5.1. Let Γ be a group. A quasi-homomorphism from Γ to $\mathbf{R}$ is a map $F \colon \Gamma \to \mathbf{R}$ such that there is a constant D such that for every γ_1, γ_2 in Γ we have $|F(\gamma_1\gamma_2) - F(\gamma_1) - F(\gamma_2)| \leq D$.

The following is an easy exercise left to the reader.

LEMMA 5.2. *Let $F \colon \mathbf{Z} \to \mathbf{R}$ be a quasi-homomorphism. Then, there exists a unique real number τ such that the sequence $F(n) - n\tau$ is bounded.*

As we shall see later, this lemma is far from being true if we replace the group $\mathbf{Z}$ by a more general group Γ.

Let us restrict the quasi-homomorphism T to the group generated by a homeomorphism $\widetilde{f}$, *i.e.* let us consider the sequence $T\big(\widetilde{f}^{\,n}\big)$. According to the lemma, there is a unique number $\tau(\widetilde{f})$ such that $T\big(\widetilde{f}^{\,n}\big) - n\tau$ is bounded. This number $\tau(\widetilde{f})$ is by definition the *translation number* of $\widetilde{f}$. It follows from the definition that if we compose $\widetilde{f}$ with an integral translation, the translation number increases by an integer. If we consider an element f in $\text{Homeo}_+(\mathbf{S}^1)$, the translations numbers of its lifts in $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ differ by integers so that the element $\rho(f) = \tau(\widetilde{f}) \mod \mathbf{Z} \in \mathbf{R}/\mathbf{Z}$ is well defined. This is called the *rotation number* of the homeomorphism f.

These definitions show that τ is a quasi-homomorphism from $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ to $\mathbf{R}$ and that it has been "normalized" so that it is a homomorphism on each one generator subgroup, *i.e.* we have $\tau\big(\widetilde{f}^{\,n}\big) = n\tau(\widetilde{f})$ for every $\widetilde{f}$ and every integer n.

Of course, it is an easy matter to check that the translation number of the translation by τ in $\mathbf{R}$ is τ and that the rotation number of the rotation $x \in \mathbf{R}/\mathbf{Z} \mapsto x+\rho \in \mathbf{R}/\mathbf{Z}$ of "angle" ρ on the circle is indeed ρ as it should be !

The next proposition is easy but is a justification for introducing these numbers.

PROPOSITION 5.3. *The translation number and the rotation number are invariant under conjugation in* $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ *and* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *respectively.*

Proof. This follows formally from the fact that τ is a quasi-homomorphism and is a homomorphism on one generator groups. Indeed,

$$\tau\left(\widetilde{f}^{n}\right) = n\tau(\widetilde{f})$$

and

$$\tau\left(\widetilde{h}\widetilde{f}^{n}\widetilde{h}^{-1}\right) = \tau\left((\widetilde{h}\widetilde{f}\widetilde{h}^{-1})^{n}\right) = n\tau(\widetilde{h}\widetilde{f}\widetilde{h}^{-1})$$

differ by a bounded amount, independent of n, so that they must be equal. This shows that the translation number is a conjugacy invariant. The assertion concerning the rotation number follows immediately. □

Let us give some universal characterization of the translation number.

PROPOSITION 5.4 ([4]). *The translation number is the unique quasi-homomorphism* $\tau\colon \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \to \mathbf{R}$ *which is a homomorphism when restricted to one generator groups and which takes the value* 1 *on the translation by* 1.

Proof. An easy generalization of Lemma 5.2 shows that any quasi-homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ differs from a homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ by a bounded amount. This implies that if a quasi-homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ is a homomorphism when restricted to one generator groups, it is a homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ (note that a bounded homomorphism is necessarily trivial).

Let t be another quasi-homomorphism satisfying the conditions of the proposition. It follows from our first observation that t is a homomorphism when restricted to the (commutative) group generated by one element $\widetilde{f}$ and the integral translations. Consider now the difference $r = \tau - t$. Its value on an element $\widetilde{f}$ depends only on the projection of $\widetilde{f}$ in $\mathrm{Homeo}_+(\mathbf{S}^1)$ so that we get a quasi-homomorphism $\overline{r}\colon \mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}$ which is a homomorphism on one generator groups. We claim that $\overline{r}$ must be trivial. This will follow from a property of $\mathrm{Homeo}_+(\mathbf{S}^1)$ that we shall prove later (see 5.11): any homeomorphism f in $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be written as a commutator $[f_1,f_2] = f_1f_2f_1^{-1}f_2^{-1}$. (In fact we only prove in 5.11 that any homeomorphism is a product of *two* commutators but this is enough for the proof which follows.) Assuming this result, we see that any quasi-homomorphism from $\text{Homeo}_+(\mathbf{S}^1)$ has to be bounded. Indeed, up to a bounded amount, the value of the quasi-homomorphism $\bar{r}$ on $f = [f_1,f_2]$ is equal to the sum of its values on $f_1,f_2,f_1^{-1},f_2^{-1}$ which is bounded (since $\bar{r}(f_1)+\bar{r}(f_1^{-1})$ is bounded). Now, a bounded quasi-homomorphism which is a homomorphism on one generator groups is trivial so that $\bar{r}$ is zero. □

We mention a very interesting problem coming from [37]:

PROBLEM 5.5 (Jankins-Neumann). *Let $\mathcal{R} \subset (\mathbf{R}/\mathbf{Z})^3$ be the set of triples (ρ_1, ρ_2, ρ_3) such that there exist three elements f_1, f_2, f_3 of* $\text{Homeo}_+(\mathbf{S}^1)$ *such that $f_1f_2f_3 = \text{Id}$ and whose rotation numbers are (ρ_1, ρ_2, ρ_3). Can one describe this set $\mathcal{R}$ explicitly?*

In [37], the authors show that $\mathcal{R}$ has a fractal structure. First, they explicitly describe the set $\mathcal{R}_0 \subset (\mathbf{R}/\mathbf{Z})^3$ of triples (ρ_1, ρ_2, ρ_3) such that there exist three elements f_1, f_2, f_3 of some $\text{PSL}_k(2,\mathbf{R})$ such that $f_1f_2f_3 = \text{Id}$ and whose rotation numbers are (ρ_1, ρ_2, ρ_3). Of course, $\mathcal{R}_0 \subset \mathcal{R}$ and they conjecture that these two sets are equal. As a motivation for their conjecture, they find an explicit set $\mathcal{R}_1$ such that $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_1$ and such that $\mathcal{R}_1 - \mathcal{R}_0$ is "small": the Lebesgue measure of $\mathcal{R}_1 - \mathcal{R}_0$ is indeed $0.0010547\ldots$ and the Lebesgue measure of $\mathcal{R}_0$ is $25/8+3\zeta(2)+3\zeta(3)-6\zeta(2)\zeta(3)/\zeta(5) \simeq 0.224649208402\ldots$ (where ζ is the Riemann ζ-function). As Jankins and Neumann write, their conjecture is therefore 99.9 % proved!

We shall show that the "number" $\rho(f)$ contains a lot of information on the topological dynamics of f. Let us begin by explaining the main possibilities for the dynamics of an arbitrary group of homeomorphisms.

PROPOSITION 5.6. *Let Γ be any subgroup of* $\text{Homeo}_+(\mathbf{S}^1)$. *Then there are three mutually exclusive possibilities.*

1) *There is a finite orbit.*

2) *All orbits are dense.*

3) *There is a compact Γ-invariant subset $K \subset \mathbf{S}^1$ which is infinite and different from $\mathbf{S}^1$ and such that the orbits of points in K are dense in K. This set K is unique, contained in the closure of any orbit and is homeomorphic to a Cantor set.*

Proof. Let us consider the collection of compact sets in $\mathbf{S}^1$ which are non empty and Γ-invariant, ordered by inclusion. By Zorn's lemma, there is a minimal set in this collection. Choose such a *minimal set* K. Note that the closure of the orbit of any point in K is a closed non empty Γ-invariant set contained in K so that it must coincide with K by minimality: the orbit of a point in K is dense in K. Observe now that the topological boundary $\partial K = K - interior(K)$ and the set K' of accumulation points of K are closed and Γ-invariant. Hence, we have the following possibilities.

1) K' is empty. In this case, K is finite and we found a finite orbit.

2) ∂K is empty, so that K is the full circle. In this case, all orbits are dense.

3) $K' = K$ and $\partial K = K$, so that K is a compact perfect set in the circle with empty interior: this is one definition of a Cantor set.

In order to prove the uniqueness of K in the last case, we show that K is contained in the closure of any orbit. The complement of K in the circle is a disjoint union of a countable family of open intervals. Let x be a point in the complement of K, lying in some interval I and let a be the origin of I (note that I is oriented). Finally, let y be any point in K. Since we know that the orbit of any point of K is dense in K and that K has no isolated point, there is a sequence of elements γ_n such that $\gamma_n(a)$ consists of distinct points and converges to y. The intervals $\gamma_n(I)$ are therefore disjoint so that the distance between $\gamma_n(a)$ and $\gamma_n(x)$ converges to zero. It follows that $\gamma_n(x)$ converges to y. This proves that K is contained in the closure of every orbit and the uniqueness of the minimal set K follows immediately. □

Case 3 looks strange at first sight: it is called the *exceptional minimal set* case for this reason. We reduce it to case 2, using the notion of *semi-conjugacy*. Consider a map $\widetilde{h}$ from $\mathbf{R}$ to $\mathbf{R}$ which is continuous, increasing (if $x \leq y$ then $\widetilde{h}(x) \leq \widetilde{h}(y)$) and which commutes with integral translations. We stress the fact that $\widetilde{h}$ might be non injective: typically it might be constant on some intervals. Such a map defines a map h from the circle to itself. We call such a map an *increasing continuous map of degree* 1 *from the circle to itself.*

DEFINITION 5.7. Let Γ be a group and ϕ_1, ϕ_2 be two homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. We say that ϕ_1 is *semi-conjugate* to ϕ_2 if there is an increasing continuous map h of degree 1 from the circle to itself such that for every γ in Γ, we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

Observe that this notion is not symmetric: ϕ_2 is not necessarily semi-conjugate to ϕ_1.

PROPOSITION 5.8. *Let* Γ *be a group and* ϕ *be a homomorphism from* Γ *to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that* $\phi(\Gamma)$ *has an exceptional minimal set* K. *Then there is a homomorphism* $\overline{\phi}$ *from* Γ *to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that* ϕ *is semi-conjugate to* $\overline{\phi}$ *and* $\overline{\phi}(\Gamma)$ *has dense orbits on the circle.*

Proof. The complement of K in the circle is a countable union of open intervals. For each of these intervals, collapse its closure to a point. The resulting quotient space is homeomorphic to a circle. In other words, there is an increasing continuous map h of degree 1 from the circle to itself such that $h(K) = \mathbf{S}^1$ and such that the fibers $h^{-1}(x)$ are either points or the closed intervals which are the closures of the connected components of the complement of K. Since $\phi(\Gamma)$ acts on the circle and preserves K, it also acts on the "collapsed" circle so that we can define another homomorphism $\overline{\phi}$ which satisfies the conditions of the proposition (we know that orbits of points in K are dense in K). □

The main object of these notes is to discuss the dynamics of "big groups" Γ acting on the circle. However, we first restrict ourselves to the "easy" case where Γ is generated by one element so that we really study the dynamics of one homeomorphism of the circle. Of course, we allow ourselves to say that a homeomorphism f_1 is semi-conjugate to f_2 if the corresponding homomorphisms from $\mathbf{Z}$ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ are semi-conjugate. The following result shows that the rotation number of a homeomorphism contains a lot of information on the dynamics.

THEOREM 5.9 (Poincaré). *Let* f *be an element of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Then* f *has a periodic orbit if and only if the rotation number* $\rho(f)$ *is rational, i.e. belongs to* $\mathbf{Q}/\mathbf{Z}$. *If the rotation number* $\rho(f)$ *is irrational, then* f *is semi-conjugate to the rotation on the circle of angle* $\rho(f) \in \mathbf{R}/\mathbf{Z}$. *This semi-conjugacy is actually a conjugacy if the orbits of* f *are dense.*

Proof. Choose a lift $\widetilde{f}$ of f in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. We know that the numbers $\widetilde{f}^n(x) - n\tau(\widetilde{f})$ are uniformly bounded (independently of $n \in \mathbf{Z}$). Define $\widetilde{h}(x) = \sup_n \big(\widetilde{f}^n(x) - n\tau(\widetilde{f})\big)$. The following properties of $\widetilde{h}$ are obvious:

1) $\widetilde{h}$ is increasing (it is left continuous but not necessarily continuous).
2) $\widetilde{h}(x+1) = \widetilde{h}(x) + 1$.
3) $\widetilde{h}(\widetilde{f}(x)) = \widetilde{h}(x) + \tau(\widetilde{f})$.

If $\tilde{h}$ were continuous, that would lead to a semi-conjugacy between f and the rotation by an angle $\tau(\tilde{f})$ mod $\mathbf{Z} = \rho(f) \in \mathbf{R}/\mathbf{Z}$. The structure of an increasing function like $\tilde{h}$ from $\mathbf{R}$ to $\mathbf{R}$ is not difficult to analyze. First, the fibers $\tilde{h}^{-1}(x)$ are either points or intervals. There is at most a countable number of these intervals: call the union of the interior of these intervals the plateau set $Plat(\tilde{h})$ of $\tilde{h}$; it is empty if and only if $\tilde{h}$ is injective. Second, the image $\tilde{h}(\mathbf{R})$ is the complement of the union of at most countably many disjoint intervals: call the union of the interior of these intervals the jump set $Jump(\tilde{h})$ of $\tilde{h}$; it is empty if and only if $\tilde{h}$ is continuous and onto. In our situation it is clear from 2 that $Plat(\tilde{h})$ and $Jump(\tilde{h})$ are open sets which are invariant under integral translations, so that they define open sets in the circle. Moreover, property 3 shows that $Plat(\tilde{h})$ is invariant under $\tilde{f}$ and $Jump(\tilde{h})$ is invariant under the translation by $\tau(\tilde{f})$ so that the corresponding open sets in the circle are invariant under f and the rotation by an angle $\rho(f)$ respectively.

We can now prove the theorem. Assume first that $\tau(\tilde{f})$ is irrational so that all the orbits of the rotation of angle $\rho(f)$ are dense. It follows that $Jump(\tilde{h})$ has to be empty so that $\tilde{h}$ defines a semi-conjugacy between f and the rotation by the angle $\rho(f)$. If the orbits of f are dense, then $Plat(\tilde{h})$ has to be empty and $\tilde{h}$ defines an actual conjugacy between f and the rotation. Also, since a semi-conjugacy maps finite orbit to finite orbit, f cannot have any periodic orbits in the case $\rho(f)$ irrational.

Assume that $\tau(\tilde{f})$ is a rational number of the form p/q. The we know that the element $\tilde{l}$ defined by $\tilde{l}(x) = \tilde{f}^q - p$ has a vanishing translation number so that the orbit of any point x in $\mathbf{R}$ under $\tilde{l}$ is bounded. The upper bound of any orbit is a fixed point of $\tilde{l}$. Since $\tilde{l}$ projects to f^q in $\mathrm{Homeo}_+(\mathbf{S}^1)$, we have found a fixed point for f^q, hence a periodic orbit for f. This establishes the theorem. Note that in this last case, we showed something more: if $\rho(f) = p/q$ mod $\mathbf{Z}$, then there is a periodic orbit whose "cyclic ordering" is the same as a rotation of angle p/q. This means that there is a homeomorphism h in $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose *restriction* to the periodic orbit is a conjugacy between f and the rotation of angle p/q. □

We can now describe the dynamics of a homeomorphism f in $\mathrm{Homeo}_+(\mathbf{S}^1)$ quite precisely.

Suppose first that $\rho(f)$ is *irrational*: we have two possibilities.

1) If all orbits are dense, then f is conjugate to the rotation of angle $\rho(f)$.

2) If there is an exceptional minimal set $K \subset \mathbf{S}^1$ then f is semi-conjugate to the rotation of angle $\rho(f)$. The connected components of the complement of K are wandering intervals, *i.e.* disjoint from all their iterates.

It is not difficult to construct examples of the second type. Start with an irrational rotation of angle ρ on the circle and choose a (dense) orbit $\mathcal{O} \subset \mathbf{S}^1$. Then "blow up" each point in $\mathcal{O}$ to replace it by an interval. In other words, consider a continuous increasing map h of degree 1 such that $h^{-1}(x)$ is an interval if x is in $\mathcal{O}$ and a point otherwise. The complement of the interior of these intervals is a Cantor set K. Then we construct a homeomorphism f of the circle which preserves K. On K the homeomorphism f is uniquely defined by the fact that h is a semi-conjugacy with the rotation. On the intervals of the complement of K, there is still some freedom in the construction: we choose any homeomorphism f which sends the interval $h^{-1}(x)$ to the interval $h^{-1}(x+\rho)$ for x in the orbit $\mathcal{O}$. The problem with this construction is that it is not clear whether or not we might do it in such a way that the corresponding homeomorphism f is smooth. Poincaré thought that there could exist an example of type 2 for which f is a real analytic diffeomorphism [59]: he was wrong, as shown later by Denjoy ! Again, we refrain from discussing this point here since we decided to restrict these notes to topological problems.

Suppose now that $\rho(f)$ is *rational* so that f has a periodic point. Replacing f by one of its powers f^q, we study the case where f has a fixed point. To understand the dynamics of f, we have first to describe the set $Fix(f)$ of fixed points which can be an arbitrary compact set in the circle (so that it could be rather complicated). Then, f induces a homeomorphism of each connected component of the complement of $Fix(f)$. On each component f can move points "to the right" or "to the left" and this information is the only dynamical information: it is easy to show that up to orientation preserving conjugacy, there are two kinds of fixed point free homeomorphisms of an open interval, those going to the right and to the left respectively.

Summing up, we have a complete description of conjugacy classes of homeomorphisms of the circle. To give a complete list of invariants is possible but not very pleasant: for instance in the case of vanishing rotation number, we should describe a compact set up to homeomorphism and labels "left" or "right" on each component of the complement.

As a corollary, we get a description of those elements of $\mathrm{Homeo}_+(\mathbf{S}^1)$ which have the form ϕ^1 for some topological flow ϕ^t on the circle. This follows immediately from our description of homeomorphisms and the description that we gave earlier of topological flows.

PROPOSITION 5.10. *An element f of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *can be included in a topological flow if and only if $\rho(f) = 0$ or f is conjugate to a rotation.*

Note that it is possible to find elements f which are not included in flows arbitrarily close to the identity.

We can now prove an important fact that we have already used in the proof of the simplicity of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

PROPOSITION 5.11. *Every element of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *can be written as a product of two commutators.*

Proof. Consider a topological flow on the closed interval, *i.e.* a continuous homomorphism $t \in \mathbf{R} \mapsto \phi^t \in \mathrm{Homeo}_+([0,1])$. Assume that for $t > 0$ the homeomorphism ϕ^t satisfies $\phi^t(x) > x$ for $x \in]0,1[$. By the previous discussion, all homeomorphisms ϕ^t with $t > 0$ are conjugate in $\mathrm{Homeo}_+([0,1])$. In particular, there is a homeomorphism l in $\mathrm{Homeo}_+([0,1])$ such that $l\phi^2 l^{-1} = \phi^1$. It follows that $\phi^1 = \phi^2(\phi^1)^{-1} = \phi^2 l(\phi^2)^{-1} l^{-1}$. This shows that ϕ^1 is the commutator of ϕ^2 and l. Since we know that every homeomorphism of $[0,1]$ which fixes only 0 and 1 is conjugate to ϕ^1 or its inverse, it follows that any such homeomorphism can be written as a commutator.

We described the dynamics of homeomorphisms with rotation number $0 \in \mathbf{R}/\mathbf{Z}$: in each connected component of the complement of their non empty fixed point set, they are described by a homeomorphism of the closed interval with no fixed point in the interior. Our discussion therefore implies that every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with rotation number 0 can be written as a commutator.

Consider finally an element f of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Clearly, one can choose a rotation r_θ such that fr_θ has a fixed point. In order to prove the proposition, it is therefore enough to show that any rotation can be written as a commutator. We show that this is indeed the case in $\mathrm{PSL}(2,\mathbf{R})$ using some hyperbolic geometry (of course, we could also prove the same thing by direct calculations).

Let a, b, c, d be four points in the Poincaré disc whose hyperbolic distances satisfy $dist(a,b) = dist(c,d)$ and $dist(a,d) = dist(b,c)$. Let A (resp. B) be the orientation preserving isometry of the Poincaré disc such that $A(a) = b$, $A(d) = c$ (resp. $B(a) = d$, $B(b) = c$). The commutator $ABA^{-1}B^{-1}$ fixes the point c: it is therefore a hyperbolic rotation centered at the point c. Figure 5 shows that the angle of this rotation is equal to 2π minus the sum of the angles of the quadrangle $R = abcd$ which is equal to the area of this quadrangle and can take any value between 0 and 2π since it is built out of two hyperbolic triangles. Hence we may realize any rotation as a single commutator. □

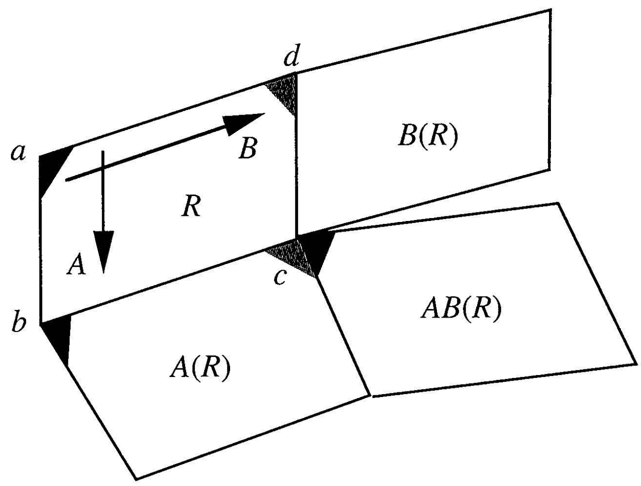

FIGURE 5

It turns out that Proposition 5.11 can be improved: every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be written as a single commutator. This is a special case of a result of [17] that we shall mention later in 6.2.

To conclude, we give some examples. Consider an element A of $\mathrm{PSL}(2,\mathbf{R})$ as a homeomorphism of the circle. The topological dynamics of A are easy to describe. Note that since A is a 2×2 matrix up to sign, the absolute value of the trace of A is well defined. If $|tr(A)| > 2$, then A is called hyperbolic and has two fixed points on the circle. In this case, the rotation number of A is of course 0. If $|tr(A)| = 2$, then A is called parabolic and has only one fixed point; its rotation number is again 0. Finally, if $|tr(A)| < 2$, then A is called elliptic and is conjugate to (the equivalence class of) a rotation matrix by some angle $2\pi\theta$ where $\theta \in \mathbf{R}/\mathbf{Z}$ is such that $2\cos(\theta) = |tr(A)|$. In this case, the rotation number is therefore $\cos^{-1}(tr(A)/2)/2\pi$.

Let us consider a finitely generated fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbf{R})$. Since it is a discrete subgroup, any elliptic element in Γ must be of finite order. Assume that Γ is torsion free. (A theorem of Selberg guarantees that any finitely generated subgroup of a matrix group contains a finite index torsion free subgroup.) Then any element of Γ has rotation number equal to 0. However, there are many fuchsian groups exhibiting very rich dynamics, even with dense orbits. These examples show that *the data of all individual rotation numbers of the elements of a group acting on the circle is far from sufficient to describe the dynamics of the group*. In other words, Theorem 5.9 cannot be generalized so easily to groups more complicated than $\mathbf{Z}$. In the next section, we shall define a more subtle invariant suitable for bigger group actions, like fuchsian groups.

Observe that our computations in $\mathrm{PSL}(2,\mathbf{R})$ show that the rotation number is a continuous function on $\mathrm{PSL}(2, \mathbf{R})$ but definitely not a smooth function. On the group $\mathrm{Homeo}_+(\mathbf{S}^1)$, we have the following behaviour:

PROPOSITION 5.12. *The map* ρ: $\mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}/\mathbf{Z}$ *is continuous and the pre-image of* $\mathbf{Q}/\mathbf{Z}$ *contains an open and dense set.*

Proof. The continuity follows immediately from the definitions. Indeed, the continuous function $\widetilde{f} \mapsto \widetilde{f}^n(0)/n$ on $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ differs at most by $1/n$ from $\tau(\widetilde{f})$. This implies the continuity of the translation number.

Suppose that $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is such that $\widetilde{f}(x)-x$ achieves both positive and negative values. Then $\widetilde{f}$ has at least a fixed point and $\tau(\widetilde{f}) = 0$. Since this condition is obviously open in the uniform topology, we have found an open set on which the translation number takes the value 0. In the same manner, we construct open sets on which τ takes the value p/q: the set of those $\widetilde{f}$ for which $\widetilde{f}^q(x) - x - p$ takes both positive and negative values.

We leave to the reader the (easy) proof that the set of $\widetilde{f}$ for which $\tau(\widetilde{f})$ is rational is dense in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. □

The local structure of the map ρ is quite interesting as was shown by a very nice example due to Arnold [1]. Consider the 2-parameter family of elements of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ given by

$$\widetilde{f}_{\alpha,\epsilon}(x) = x + \alpha + \epsilon \sin(2\pi x).$$

Here α is a real number and ϵ is a real number which is small enough to guarantee that $\widetilde{f}_{\alpha,\epsilon}$ is a homeomorphism ($|\epsilon| < 1/2\pi$ is enough). We should think of these $\widetilde{f}_{\alpha,\epsilon}$ as a small deformation of the translation by α depending of the small parameter ϵ. Let us look at the behaviour of $\tau(\widetilde{f}_{\alpha,\epsilon})$ as a function of α and ϵ. Of course, we have $\tau(\widetilde{f}_{\alpha,0}) = \alpha$. We can check rather easily the following facts. For each ϵ, the function $\alpha \mapsto \tau(\widetilde{f}_{\alpha,\epsilon})$ is continuous and increasing but is not strictly increasing for $\epsilon \neq 0$. The plateau set of this function is the complement of a Cantor set on which τ takes irrational values. The interior of the set of (α, ϵ) on which τ takes the rational value p/q is an "Arnold tongue" which touches the axis $\epsilon = 0$ at the point $(p/q, 0)$. The bigger the denominator q, the thinner the corresponding tongue.

Another interesting feature of this picture is that the Lebesgue measure of the set of (α, ϵ) for which τ is irrational is not 0. Hence, the translation number takes rational values on an open dense set but takes irrational values on a set of positive Lebesgue measure.

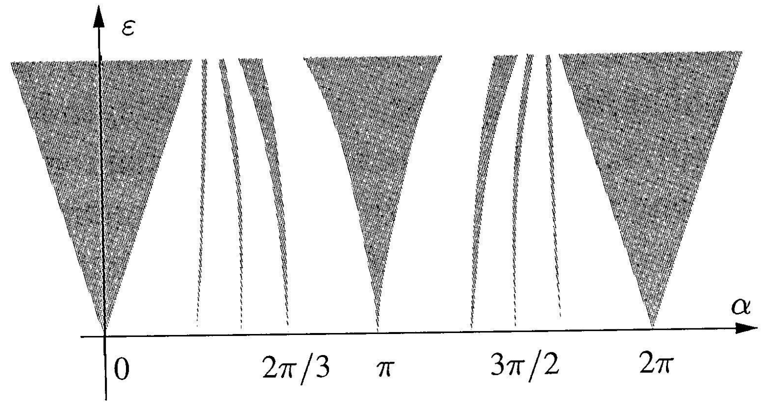

FIGURE 6

As an additional example, consider the case of piecewise linear homeomorphisms of the circle. Since the group $\mathrm{PL}_+(\mathbf{S}^1)$ contains all rotations, it is clear that the rotation number of such a homeomorphism can be arbitrary. However, it is shown in [28] that the rotation number of any element of the Thompson group G is rational and that any rational number is achieved. The proof is very indirect and there is a need for a better proof. We could formulate the problem in the following way.

PROBLEM 5.13. *Consider a rational piecewise linear homeomorphism f of the circle, i.e. such that all its slopes are rational and such that all "break-points" are rational. Is it true that the rotation number of f is rational?*

We can in fact generalize Thompson's group quite a lot in the following way. Let $\Lambda \subset \mathbf{R}_\star^+$ be a subgroup of the multiplicative group of positive real numbers and let $W \subset \mathbf{R}$ be an additive subgroup invariant under multiplication by Λ. Then we can consider the subgroup $\widetilde{G}_{\Lambda,W}$ of $\widetilde{\mathrm{PL}}_+(\mathbf{S}^1)$ consisting of those elements with slopes in Λ and break-points in W (for instance, Thompson group is the case when Λ consists of powers of 2 and W of dyadic rationals). These groups are quite interesting especially when Λ is finitely generated (see [8, 9, 63]). It would be very useful to understand the nature of translation numbers of elements of $\widetilde{G}_{\Lambda,W}$ for specific Λ and W.

In [34], one can find (among other things!) a very interesting analysis of the rotation numbers of an explicit 1-parameter family of piecewise linear homeomorphisms of the circle.

5.2 TITS' ALTERNATIVE

Recall that J. Tits proved a remarkable alternative for finitely generated subgroups Γ of $\mathrm{GL}(n, \mathbf{C})$ (see [65]): either Γ *contains a non abelian free subgroup or* Γ *contains a subgroup of finite index which is solvable.* Such an alternative does not hold for subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Indeed, we have seen that the group $\mathrm{PL}_+([0,1])$ can be considered as a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and contains no free non abelian subgroup. The subgroup F of $\mathrm{PL}_+([0,1])$ consisting of elements whose slopes are powers of 2 and whose break-points are dyadic rationals, is a finitely presented group and is certainly not virtually solvable (since its first commutator subgroup is a simple group, see [28]). However, answering a question of the author, Margulis recently proved the following theorem [49]:

THEOREM 5.14 (Margulis). *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *At least one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- *There is a probability measure on the circle which is* Γ*-invariant.*

COROLLARY 5.15. *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense in the circle. Exactly one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- Γ *is abelian and is conjugate to a group of rotations.*

The corollary follows easily from the theorem. Indeed, if all Γ-orbits are dense, any invariant probability must have full support and cannot have any non trivial atom. Any such probability is the image of the Lebesgue measure by some homeomorphism of the circle. Hence, up to some conjugacy, one can assume that Γ preserves the Lebesgue measure, *i.e.* consists of rotations. Note however that the proof which follows will begin with a proof of the corollary...

The proof of Margulis' theorem is very elegant and we cannot refrain from giving an account of it. Our presentation is a variation (or maybe a simplification ?) of Margulis' original ideas. More precise results may be found in the recent preprint [6]. We begin by recalling the "ping-pong" lemma, which is the standard way of constructing free subgroups (see [31]). Suppose a set X contains two disjoint non empty subsets A and A'. Let f, f' be two bijections of X which are such that for every $n \in \mathbf{Z} \setminus \{0\}$, we have $f^n(A) \subset A'$ and $f'^n(A') \subset A$. Then we claim that *f and f' generate a free subgroup of the group of bijections of X*. The proof is easy; consider a word $w(f,f') = f^{m_1} f'^{m'_1} \cdots f^{m_k} f'^{m'_k}$ with non zero exponents m_i, m'_i, except maybe the first one m_1 and the last one m'_k (if $k = 1$, we assume that m_1 and m'_1 are not both zero...). We want to show that $w(f,f')$ represents a non trivial bijection of X. This is clear if $m_1 \neq 0$ and $m'_k = 0$ (resp. $m_1 = 0$ and $m'_k \neq 0$) since in this case we have $w(f,f')(A) \subset A'$ (resp. $w(f,f')(A') \subset A$). In the other cases, one can conjugate $w(f,f')$ by a suitable power of f or f' to get a new word which is in the previous form. This proves the ping-pong lemma.

In the case of the circle, the typical application of the ping-pong lemma is the following. Let I, J, I', J' be four closed intervals in the circle and let f, f' be two orientation preserving homeomorphisms of the circle. Assume the following condition holds:

(PING-PONG) The four intervals I, J, I', J' are disjoint, $f'(I) = \mathbf{S}^1 \setminus interior(J)$ and $f(I') = \mathbf{S}^1 \setminus interior(J')$.

Clearly, if one sets $X = \mathbf{S}^1$, $A = I \cup J$ and $A' = I' \cup J'$, we are in the situation of the ping-pong lemma and one can deduce from (PING-PONG) that f and f' generate a free subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

In order to find free subgroups inside a given subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we shall try to locate such ping-pong situations.

Assume now that we are given a group Γ such that the following two properties hold:

(MINIMALITY) All Γ-orbits are dense.

(STRONG EXPANSIVITY) There is a sequence of closed intervals I_n in the circle and a sequence γ_n of elements of Γ such that the length of I_n tends to zero as well as the length of the complementary intervals $J_n = \mathbf{S}^1 \setminus int(\gamma_n(I_n))$.

Of course, using subsequences we can assume in (STRONG EXPANSIVITY) that both endpoints of I_n converge to some point x and that both endpoints of J_n converge to some point y. We can also assume that $x \neq y$, since otherwise we could replace γ_n by $\gamma\gamma_n$ where γ is some element of Γ such that $y = \gamma(x) \neq x$.

Choose some γ in Γ such that $x' = \gamma(x)$ and $y' = \gamma(y)$ are both different from x and y (exercise: show that such an element γ exists!) and consider the sequence $\gamma'_n = \gamma^{-1}\gamma_n\gamma$. Of course, if we let $I'_n = \gamma(I_n)$ (resp. $J'_n = \gamma(J_n)$), the sequence of intervals I'_n (resp. J'_n) shrinks to x' (resp. to y'). Clearly, if n is big enough, the four intervals $I = I_n, J = J_n, I' = I'_n, J' = J'_n$ and the two homeomorphisms $f = \gamma_n, f' = \gamma'_n$ satisfy (PING-PONG) and therefore γ_n

and γ_n' generate a free subgroup of Γ. *In other words, if* (MINIMALITY) *and* (STRONG EXPANSIVITY) *hold, then* Γ *contains a free non abelian subgroup.*

The minimality condition is not so restrictive: we saw earlier that any action without a finite orbit is semi-conjugate to such a minimal action. However, the strong expansivity condition is very restrictive. Let us introduce the following weaker condition.

(EXPANSIVITY) There is a sequence of closed intervals I_n and a sequence of elements γ_n of Γ such that the length of I_n tends to zero and the length of $\gamma_n(I_n)$ is bounded away from zero.

Call a closed interval K in the circle *contractible* if there is a sequence of elements γ_n of Γ such that the length of $\gamma_n(K)$ tends to zero. It follows from (EXPANSIVITY) that there exists a non trivial contractible interval. If moreover the condition (MINIMALITY) is also satisfied, then every point of the circle belongs to the interior of some contractible interval. So let us assume now that (MINIMALITY) and (EXPANSIVITY) are satisfied.

For each point x in the circle, consider the set of points y such that the interval $[x,y]$ is contractible. Denote by $\theta(x)$ the least upper bound of those points y (to be correct, one should lift everything to the universal cover). In this way, we get a map θ from the circle to itself. Note that obviously θ commutes with all elements of Γ. Note also that θ is monotone. We claim that θ is a homeomorphism. Indeed if it were not strictly monotone, the union *Plat*(θ) of the interiors of the intervals in which θ is constant would be a Γ-invariant open set. By (MINIMALITY), this open set is empty unless θ is constant, but this is of course not possible since this constant would be fixed by Γ. In the same way, one shows that θ is continuous, using the union *Jump*(θ) of the interiors of the "jump intervals" like in 3.2.

We now consider the rotation number of θ. If this rotation number is irrational, then θ has to be conjugate to an irrational rotation since otherwise its unique invariant minimal set would be a non trivial Γ-invariant compact set. Since a homeomorphism which commutes with an irrational rotation is itself a rotation, that would imply that Γ is conjugate to a group of rotations. This is in contradiction with (EXPANSIVITY).

Hence the rotation number of θ is rational. The union of periodic points of θ is a non empty closed set which is Γ-invariant. It follows that θ is a periodic homeomorphism.

Consider the quotient $\mathbf{S}^{1'} = \mathbf{S}^1/\theta$ of the circle by the finite cyclic group generated by θ. This is a ("shorter") circle on which we have a natural action of Γ since, once again, Γ commutes with θ.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of Γ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

We have now proved that if (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* Γ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group Γ. If a group Γ acts equicontinuously, then its closure in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

We have shown that if (MINIMALITY) *holds then* Γ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let Γ be any subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If Γ is minimal, we have already proved the theorem. If Γ has a finite orbit, there is a Γ-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the Γ-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that Γ contains a non abelian free subgroup unless the restriction of the action of Γ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a Γ-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

6. BOUNDED EULER CLASS

6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let Γ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of Γ and for which n-simplices are all $(n+1)$-tuples of elements of Γ. The i^{th} face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term γ_i is omitted. Note that the set $E\Gamma$ does not depend on the group structure of Γ.