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*fuchsian groups* which are by definition the discrete subgroups of $\mathrm{PSL}(2,\mathbf{R})$. These groups come from many parts of mathematics, in particular from number theory. For instance, the modular group $\mathrm{PSL}(2,\mathbf{Z})$ is fundamental in the study of quadratic forms in two variables over the integers and its action on $\mathbf{RP}^1$ or on $\mathcal{H}$ is one of the main tools to understand it. Gauss began its analysis in his famous *Disquisitiones* and the modular group might be the first non-commutative group to have been been studied in the history of mathematics. As another example, consider a quadratic form in three variables with integral coefficients and signature $(+,+,-)$; the group of its isometries with integer coefficients is of course a fuchsian group. This was another motivation for Poincaré when he studied these groups [60]. We also want to emphasize that not only the discrete groups of $\mathrm{PSL}(2,\mathbf{R})$ might be interesting, even from the number theoretical point of view. Examples can be given by taking a number field $k$ embedded in $\mathbf{R}$ and looking at the ring of integers $\mathcal{O}$ in this field (for instance $\mathbf{Z}[\sqrt{2}]$ in $\mathbf{Q}(\sqrt{2})$). The group $\mathrm{PSL}(2,\mathcal{O})$ of elements of $\mathrm{PSL}(2,\mathbf{R})$ with entries in $\mathcal{O}$ is a very important one (even though it is dense in $\mathrm{PSL}(2,\mathbf{R})$ if $k$ is not the field of rational numbers).

## 3.2 Piecewise linear groups

Our second example is a much bigger group: the group of piecewise linear homeomorphisms of the circle $\mathbf{S}^1$, considered here as $\mathbf{R}/\mathbf{Z}$. A homeomorphism $f$ of the real line $\mathbf{R}$ is called *piecewise linear* if there is an increasing sequence of real numbers $x_i$ parametrized by $i \in \mathbf{Z}$ such that $\lim_{\pm\infty} x_i = \pm\infty$ and such that the restriction of $f$ to each interval $[x_i, x_{i+1}]$ coincides with an affine map. If such a homeomorphism satisfies $f(x+1)=f(x)+1$ for all $x$, then it induces a homeomorphism of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. Such a homeomorphism of $\mathbf{S}^1$ is called a piecewise linear homeomorphism of the circle. Note that, by our definition, we are only considering orientation preserving homeomorphisms of the circle. The collection of these homeomorphisms is a group, denoted by $\mathrm{PL}_+(\mathbf{S}^1)$.

Again, this group is acting transitively on the circle so there is not much to say about its orbits... However $\mathrm{PL}_+(\mathbf{S}^1)$ contains some very interesting subgroups which will provide good examples of some dynamical phenomena on the circle. We shall mention only one of them.

The *Thompson group*, denoted by $G$, is a countable subgroup of $\mathrm{PL}_+(\mathbf{S}^1)$ which has been studied quite a lot recently and deserves more attention. Some of its properties will be mentioned in these notes, in particular as a source of (counter)-examples. To define it, we consider first the group $\widetilde{G}$ consisting of piecewise linear homeomorphisms $f$ of $\mathbf{R}$ which have the following four properties.

- The sequence $x_i$ can be chosen in such a way that $x_i$ and $f(x_i)$ consist of dyadic rational numbers (*i.e.* of the form $p2^q$, $p, q \in \mathbf{Z}$).
- The set of dyadic rational numbers is preserved by $f$.
- The derivatives of the restrictions of $f$ to $]x_i, x_{i+1}[$ are powers of 2 (*i.e.* of the form $2^q$, $q \in \mathbf{Z}$).
- One has $f(x+1) = f(x) + 1$ for all $x$.

The elements of $\widetilde{G}$ induce homeomorphisms of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. The collection of these homeomorphisms is the Thompson group $G$. Figure 4 shows the graphs of two typical elements of $G$.

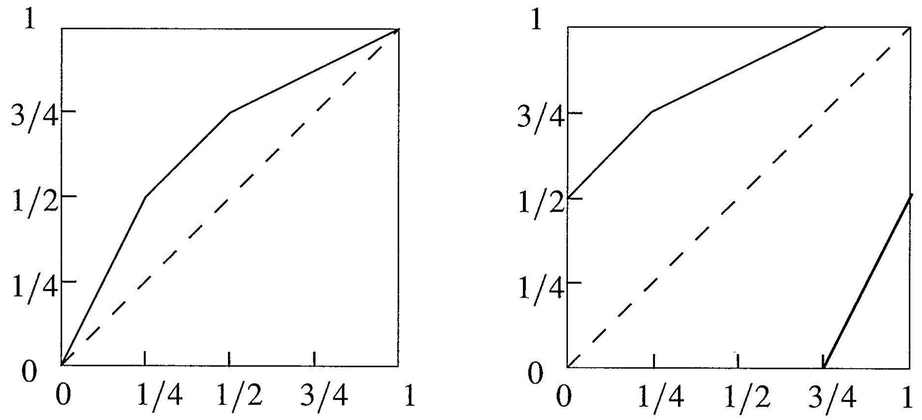


FIGURE 4

Among the nice properties of $G$, we mention first the fact that $G$ is an *infinite finitely presented simple group*. This was the main motivation for Thompson: indeed $G$ was the first example of such a group (recall that a group is called simple if it contains no proper normal subgroup).

We also mention a connection with the modular group $\mathrm{PSL}(2, \mathbf{Z})$ acting on $\mathbf{RP}^1$. Consider the group of homeomorphisms of $\mathbf{RP}^1$ which are piecewise-$\mathrm{PSL}(2, \mathbf{Z})$, *i.e.* for which one can partition $\mathbf{RP}^1$ as a finite union of intervals with rational endpoints in such a way that on each of these intervals, the homeomorphism coincides with an element of $\mathrm{PSL}(2, \mathbf{Z})$. It turns out that there is a homeomorphism $h$ from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{RP}^1$ mapping the dyadic points in $\mathbf{R}/\mathbf{Z}$ to the rational points of $\mathbf{QP}^1$ and conjugating the Thompson group $G$ with this group of piecewise-$\mathrm{PSL}(2, \mathbf{Z})$ !

Somehow, we could say that $G$ sits inside $\mathrm{PL}_+(\mathbf{S}^1)$ like a fuchsian group sits inside $\mathrm{PSL}(2, \mathbf{R})$. For more information concerning this group, see [13, 28].