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fuchsian groups which are by definition the discrete subgroups of $PSL(2, \mathbf{R})$. These groups come from many parts of mathematics, in particular from number theory. For instance, the modular group PSL(2, Z) is fundamental in the study of quadratic forms in two variables over the integers and its action on RP1 or on $\mathcal H$ is one of the main tools to understand it. Gauss began its analysis in his famous Disquisitiones and the modular group might be the first noncommutative group to have been been studied in the history of mathematics. As another example, consider a quadratic form in three variables with integral coefficients and signature (+,+,-); the group of its isometries with integer coefficients is of course a fuchsian group. This was another motivation for Poincaré when he studied these groups [60]. We also want to emphasize that not only the discrete groups of $PSL(2, \mathbf{R})$ might be interesting, even from the number theoretical point of view. Examples can be given by taking a number field k embedded in \mathbf{R} and looking at the ring of integers \mathcal{O} in this field (for instance $\mathbb{Z}[\sqrt{2}]$ in $\mathbb{Q}(\sqrt{2})$). The group $PSL(2, \mathcal{O})$ of elements of $PSL(2, \mathbf{R})$ with entries in \mathcal{O} is a very important one (even though it is dense in $PSL(2, \mathbf{R})$ if k is not the field of rational numbers).

3.2 PIECEWISE LINEAR GROUPS

Our second example is a much bigger group: the group of piecewise linear homeomorphisms of the circle S^1 , considered here as \mathbf{R}/\mathbf{Z} . A homeomorphism f of the real line \mathbf{R} is called *piecewise linear* if there is an increasing sequence of real numbers x_i parametrized by $i \in \mathbf{Z}$ such that $\lim_{\pm \infty} x_i = \pm \infty$ and such that the restriction of f to each interval $[x_i, x_{i+1}]$ coincides with an affine map. If such a homeomorphism satisfies f(x+1) = f(x) + 1 for all x, then it induces a homeomorphism of the circle $S^1 \simeq \mathbf{R}/\mathbf{Z}$. Such a homeomorphism of S^1 is called a piecewise linear homeomorphism of the circle. Note that, by our definition, we are only considering orientation preserving homeomorphisms of the circle. The collection of these homeomorphisms is a group, denoted by $PL_+(S^1)$.

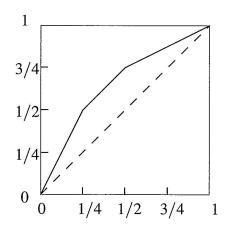
Again, this group is acting transitively on the circle so there is not much to say about its orbits... However $PL_+(S^1)$ contains some very interesting subgroups which will provide good examples of some dynamical phenomena on the circle. We shall mention only one of them.

The *Thompson group*, denoted by G, is a countable subgroup of $PL_+(S^1)$ which has been studied quite a lot recently and deserves more attention. Some of its properties will be mentioned in these notes, in particular as a source of (counter)-examples. To define it, we consider first the group \widetilde{G} consisting

of piecewise linear homeomorphisms f of \mathbf{R} which have the following four properties.

- The sequence x_i can be chosen in such a way that x_i and $f(x_i)$ consist of dyadic rational numbers (i.e. of the form $p2^q$, $p, q \in \mathbb{Z}$).
- The set of dyadic rational numbers is preserved by f.
- The derivatives of the restrictions of f to $]x_i, x_{i+1}[$ are powers of 2 (i.e. of the form $2^q, q \in \mathbb{Z}$).
- One has f(x+1) = f(x) + 1 for all x.

The elements of \widetilde{G} induce homeomorphisms of the circle $S^1 \simeq \mathbf{R}/\mathbf{Z}$. The collection of these homeomorphisms is the Thompson group G. Figure 4 shows the graphs of two typical elements of G.



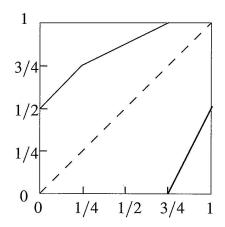


FIGURE 4

Among the nice properties of G, we mention first the fact that G is an *infinite finitely presented simple group*. This was the main motivation for Thompson: indeed G was the first example of such a group (recall that a group is called simple if it contains no proper normal subgroup).

We also mention a connection with the modular group $PSL(2, \mathbb{Z})$ acting on \mathbb{RP}^1 . Consider the group of homeomorphisms of \mathbb{RP}^1 which are piecewise- $PSL(2, \mathbb{Z})$, *i.e.* for which one can partition \mathbb{RP}^1 as a finite union of intervals with rational endpoints in such a way that on each of these intervals, the homeomorphism coincides with an element of $PSL(2, \mathbb{Z})$. It turns out that there is a homeomorphism h from \mathbb{R}/\mathbb{Z} to \mathbb{RP}^1 mapping the dyadic points in \mathbb{R}/\mathbb{Z} to the rational points of \mathbb{QP}^1 and conjugating the Thompson group G with this group of piecewise- $PSL(2, \mathbb{Z})$!

Somehow, we could say that G sits inside $PL_{+}(S^{1})$ like a fuchsian group sits inside $PSL(2, \mathbf{R})$. For more information concerning this group, see [13, 28].