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3. Two basic examples

Up to homeomorphism, the circle is the only compact connected 1-dimensional manifold: this is probably the reason why we meet so many circles in mathematics... We can think of the circle $\mathbf{S}^1$ in many ways. We can first consider it as the unit circle in $\mathbf{R}^2$ but we can also see it as the abstract 1-dimensional manifold which is the quotient of the real line $\mathbf{R}$ by the subgroup of integers $\mathbf{Z}$. From this point of view, $\mathbf{S}^1$ can be thought of as being an abelian group, isomorphic to $SO(2, \mathbf{R})$ or to the 1-dimensional torus. The circle can also be considered as the real projective line $\mathbf{RP}^1$ consisting of lines in $\mathbf{R}^2$ going through the origin (identified with $\mathbf{R} \cup \{\infty\}$ by taking the slope of a line).

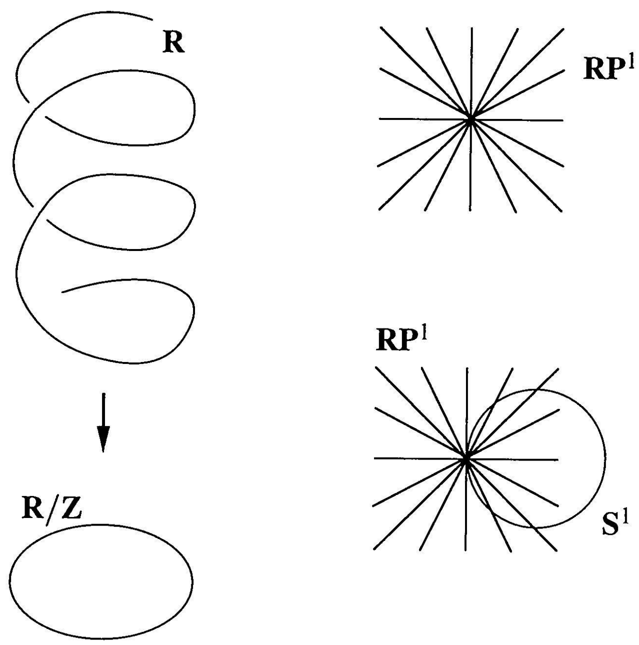

Figure 1

Going from one point of view to another is easy, the identifications being given by:

$$t \in \mathbf{R}/\mathbf{Z} \mapsto (\cos(2\pi t), \sin(2\pi t)) \in \mathbf{S}^1 \subset \mathbf{R}^2$$

$$t \in \mathbf{R}/\mathbf{Z} \mapsto \tan(\pi t) \in \mathbf{R} \cup \{\infty\} = \mathbf{RP}^1$$

$$s \in \mathbf{RP}^1 \mapsto (\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}) \in \mathbf{S}^1 .$$

In this first section, we would like to give two very basic examples of groups acting on the circle which will play a central role in these lecture notes. The properties of these examples will be detailed in this text and we could say that a major theme of research would be to show that many groups acting on the circle can be reduced to them.

3.1 The projective group

The linear group $\mathrm{GL}(2,\mathbf{R})$ consists of 2×2 real invertible matrices. Its center is the group of scalar matrices and the quotient of $\mathrm{GL}(2,\mathbf{R})$ by this center is denoted by $\mathrm{PGL}(2,\mathbf{R})$ and called the *projective group*. There is a natural (projective) action of $\mathrm{PGL}(2,\mathbf{R})$ on the circle (seen as $\mathbf{RP}^1$). Indeed, $\mathrm{GL}(2,\mathbf{R})$ acts linearly on $\mathbf{R}^2$ and induces an action on the set of lines in $\mathbf{R}^2$ going through the origin, which is $\mathbf{RP}^1$ by definition. A formula for the action is given by:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, x\right) \in \mathrm{PGL}(2,\mathbf{R}) \times \mathbf{RP}^1 \mapsto \frac{ax+b}{cx+d} \in \mathbf{RP}^1 .$$

We use a (square) bracket to denote the equivalence class modulo scalar matrices. Note that the group $\mathrm{PGL}(2,\mathbf{R})$ has two connected components given by the sign of the determinant. The component of the identity is isomorphic to $\mathrm{PSL}(2,\mathbf{R})$, which is the quotient of the unimodular group $\mathrm{SL}(2,\mathbf{R})$ by its center which consists of $\pm\mathrm{Id}$. The action of an element of $\mathrm{PGL}(2,\mathbf{R})$ on the circle preserves or reverses orientation according to the sign of its determinant.

An important feature of this action is that it extends to the disc. The real projective line $\mathbf{RP}^1$ sits naturally inside the complex projective line $\mathbf{CP}^1 \simeq \mathbf{C} \cup \{\infty\}$ which is the Riemann sphere. In the same way, the real projective group $\mathrm{PGL}(2,\mathbf{R})$ is a subgroup of the complex projective group $\mathrm{PGL}(2,\mathbf{C})$ which acts on the Riemann sphere by Mœbius transformations.

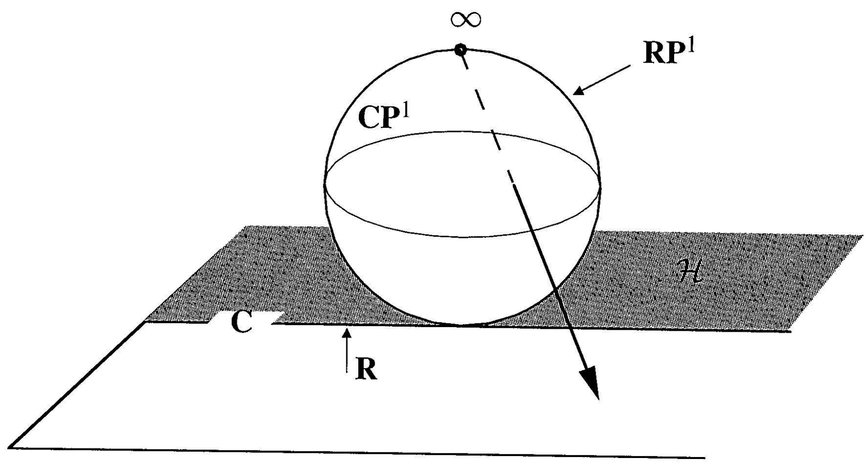

Figure 2

Hence we get an action of $\mathrm{PGL}(2,\mathbf{R})$ on the Riemann sphere $\mathbf{CP}^1$ preserving the circle $\mathbf{RP}^1$. The complement of this circle in this sphere consists of two discs which are preserved or permuted by an element of PGL(2, **R**) according to the sign of the determinant. In the obvious coordinates, the circle $\mathbf{RP}^1$ is the real axis in $\mathbf{CP}^1 \simeq \mathbf{C} \cup \{\infty\}$ plus the point at infinity. Denote by $\mathcal{H} \subset \mathbf{C} \subset \mathbf{CP}^1$ the upper half space, *i.e.* the set of complex numbers z with positive imaginary part: this is one of the two components of the complement of $\mathbf{RP}^1$ in $\mathbf{CP}^1$. We get an action of PSL(2, **R**) on $\mathcal{H}$ which extends the action of PSL(2, **R**) on the boundary $\mathbf{RP}^1$. This extension is holomorphic and is actually an isometric action when we equip $\mathcal{H}$ with its Poincaré metric (see for instance [67]).

The group PSL(2, **R**) and the rotation group SO(3, **R**) are the only simple Lie groups of real dimension 3 and there is no non trivial simple Lie group of lower dimension [58]. This may explain why several versions of these groups occur in mathematics. We give one of them, showing a different aspect of the action of PSL(2, **R**) on the circle.

Consider the quadratic form $Q = x_1^2 + x_2^2 - x_3^2$ on $\mathbf{R}^3$. Its group of isometries is denoted by O(2, 1). This group has four connected components (see for example [54]) and it turns out that the component of the identity is isomorphic to PSL(2, **R**). A simple way to check this fact is to consider the action of GL(2, **R**) on the space M(2, **R**) of 2×2 matrices given by conjugation:

$$(A, M) \in \mathrm{GL}(2, \mathbf{R}) \times \mathrm{M}(2, \mathbf{R}) \mapsto AMA^{-1}.$$

Note that this action factors through an action of PGL(2, **R**) since the center acts of course trivially. We can moreover restrict this action to the invariant 3-dimensional vector space E consisting of matrices whose trace is 0. Finally, we observe that the determinant of M provides an invariant quadratic form on E. It is easy to check that the signature of this quadratic form is $(-, -, +)$ so that, using suitable coordinates, we get an injection of PGL(2, **R**) in O(2, 1). This injection gives the promised identification between PSL(2, **R**) and the connected component of the identity in O(2, 1). Figure 3 shows the orbits of this linear action on E.

Since O(2, 1) acts linearly on $\mathbf{R}^3$, it acts projectively on the projective plane $\mathbf{RP}^2$ consisting of lines in $\mathbf{R}^3$. The zero locus of Q in $\mathbf{R}^3$ is a cone which projects to a conic C in $\mathbf{RP}^2$ invariant under O(2, 1). As any non degenerate conic in the projective plane can be rationally parametrized by $\mathbf{RP}^1$, we get an action of O(2, 1) on the circle $\mathbf{RP}^1$. The reader will easily check that we get, up to conjugacy and identifications, the same action of PSL(2, **R**) on $\mathbf{RP}^1$ that we described earlier.

The conic C bounds two domains in $\mathbf{RP}^2$. One of them is homeomorphic to a disc and is the projection of the set of points for which $Q < 0$: it

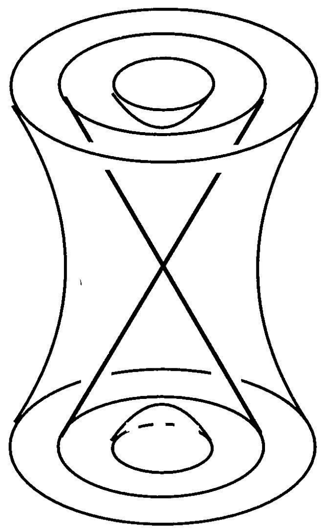

FIGURE 3

is called its interior and is denoted by D. The exterior is homeomorphic to a Mœbius band. Hence we can think of the circle $C \simeq \mathbf{RP}^1 \simeq \mathbf{S}^1$ as the boundary of a disc $D \subset \mathbf{RP}^2$ on which $\mathrm{PGL}(2, \mathbf{R})$ acts projectively. We have extended the action of $\mathrm{PGL}(2, \mathbf{R})$ on the circle to an action on the disc. This is the Klein model.

Of course, the two extensions of the action of $\mathrm{PSL}(2, \mathbf{R})$ on a disc are conjugate, even though they don't quite look the same. The first one is conformal in one complex variable and the second one is projective in two real variables. There are several ways of describing a conjugacy between these two actions [67]. The following one is nice and not so well known. Consider the linear action of $\mathrm{PSL}(2, \mathbf{R})$ on the 3-dimensional vector space of polynomials of the second degree $aX^2 + bXY + cY^2$ by linear change of coordinates. The discriminant $b^2 - 4ac$ defines an invariant quadratic form of signature $(+, +, -)$. Hence, we can identify this linear action with the linear action of the identity component of $\mathrm{O}(2, 1)$ that we considered above. Now any polynomial in the negative cone of the discriminant defines a polynomial $aX^2 + bX + c$ with two complex conjugate roots. Hence, we can define a map from the disc D to the upper half plane $\mathcal{H}$ sending the line through the polynomial to the unique root in $\mathcal{H}$. This map is obviously a conjugation between the two actions of $\mathrm{PSL}(2, \mathbf{R})$ on $\mathcal{D}$ and $\mathcal{H}$. Note however that the two actions of $\mathrm{PSL}(2, \mathbf{R})$ on $\mathbf{RP}^2$ and $\mathbf{CP}^1$ that we constructed are not conjugate since $\mathbf{RP}^2$ and $\mathbf{CP}^1$ are not homeomorphic !

The action of $\mathrm{PSL}(2, \mathbf{R})$ on the circle that we described is well known and there is not much to say about its dynamics since it has only one orbit ! In order to get examples which are interesting from the dynamical point of view, we should restrict it to suitable subgroups of $\mathrm{PSL}(2, \mathbf{R})$. We mention the *fuchsian groups* which are by definition the discrete subgroups of $\mathrm{PSL}(2,\mathbf{R})$. These groups come from many parts of mathematics, in particular from number theory. For instance, the modular group $\mathrm{PSL}(2,\mathbf{Z})$ is fundamental in the study of quadratic forms in two variables over the integers and its action on $\mathbf{RP}^1$ or on $\mathcal{H}$ is one of the main tools to understand it. Gauss began its analysis in his famous *Disquisitiones* and the modular group might be the first non-commutative group to have been been studied in the history of mathematics. As another example, consider a quadratic form in three variables with integral coefficients and signature $(+,+,-)$; the group of its isometries with integer coefficients is of course a fuchsian group. This was another motivation for Poincaré when he studied these groups [60]. We also want to emphasize that not only the discrete groups of $\mathrm{PSL}(2,\mathbf{R})$ might be interesting, even from the number theoretical point of view. Examples can be given by taking a number field k embedded in $\mathbf{R}$ and looking at the ring of integers $\mathcal{O}$ in this field (for instance $\mathbf{Z}[\sqrt{2}]$ in $\mathbf{Q}(\sqrt{2})$). The group $\mathrm{PSL}(2,\mathcal{O})$ of elements of $\mathrm{PSL}(2,\mathbf{R})$ with entries in $\mathcal{O}$ is a very important one (even though it is dense in $\mathrm{PSL}(2,\mathbf{R})$ if k is not the field of rational numbers).

3.2 Piecewise linear groups

Our second example is a much bigger group: the group of piecewise linear homeomorphisms of the circle $\mathbf{S}^1$, considered here as $\mathbf{R}/\mathbf{Z}$. A homeomorphism f of the real line $\mathbf{R}$ is called *piecewise linear* if there is an increasing sequence of real numbers x_i parametrized by $i \in \mathbf{Z}$ such that $\lim_{\pm\infty} x_i = \pm\infty$ and such that the restriction of f to each interval $[x_i, x_{i+1}]$ coincides with an affine map. If such a homeomorphism satisfies $f(x+1) = f(x)+1$ for all x, then it induces a homeomorphism of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. Such a homeomorphism of $\mathbf{S}^1$ is called a piecewise linear homeomorphism of the circle. Note that, by our definition, we are only considering orientation preserving homeomorphisms of the circle. The collection of these homeomorphisms is a group, denoted by $\mathrm{PL}_+(\mathbf{S}^1)$.

Again, this group is acting transitively on the circle so there is not much to say about its orbits… However $\mathrm{PL}_+(\mathbf{S}^1)$ contains some very interesting subgroups which will provide good examples of some dynamical phenomena on the circle. We shall mention only one of them.

The *Thompson group*, denoted by G, is a countable subgroup of $\mathrm{PL}_+(\mathbf{S}^1)$ which has been studied quite a lot recently and deserves more attention. Some of its properties will be mentioned in these notes, in particular as a source of (counter)-examples. To define it, we consider first the group $\widetilde{G}$ consisting of piecewise linear homeomorphisms f of $\mathbf{R}$ which have the following four properties.

- The sequence x_i can be chosen in such a way that x_i and $f(x_i)$ consist of dyadic rational numbers (*i.e.* of the form $p2^q$, $p,q \in \mathbf{Z}$).
- The set of dyadic rational numbers is preserved by f.
- The derivatives of the restrictions of f to $]x_i, x_{i+1}[$ are powers of 2 (*i.e.* of the form 2^q, $q \in \mathbf{Z}$).
- One has $f(x+1) = f(x) + 1$ for all x.

The elements of $\widetilde{G}$ induce homeomorphisms of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. The collection of these homeomorphisms is the Thompson group G. Figure 4 shows the graphs of two typical elements of G.

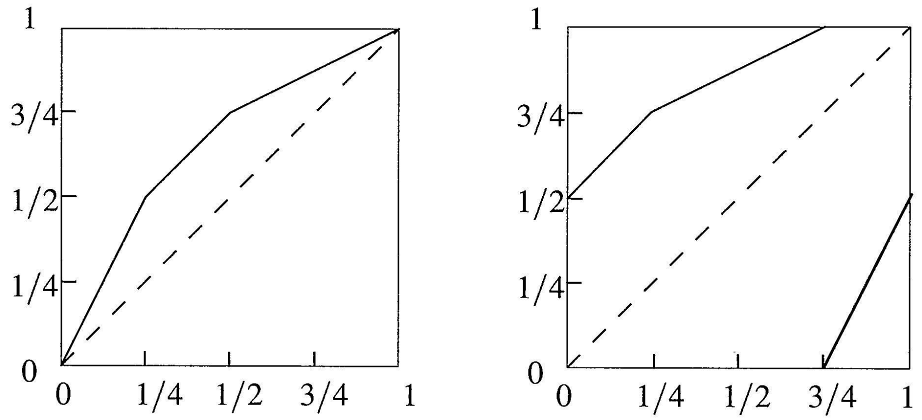

FIGURE 4

Among the nice properties of G, we mention first the fact that G is an *infinite finitely presented simple group*. This was the main motivation for Thompson: indeed G was the first example of such a group (recall that a group is called simple if it contains no proper normal subgroup).

We also mention a connection with the modular group $\mathrm{PSL}(2, \mathbf{Z})$ acting on $\mathbf{RP}^1$. Consider the group of homeomorphisms of $\mathbf{RP}^1$ which are piecewise-$\mathrm{PSL}(2, \mathbf{Z})$, *i.e.* for which one can partition $\mathbf{RP}^1$ as a finite union of intervals with rational endpoints in such a way that on each of these intervals, the homeomorphism coincides with an element of $\mathrm{PSL}(2, \mathbf{Z})$. It turns out that there is a homeomorphism h from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{RP}^1$ mapping the dyadic points in $\mathbf{R}/\mathbf{Z}$ to the rational points of $\mathbf{QP}^1$ and conjugating the Thompson group G with this group of piecewise-$\mathrm{PSL}(2, \mathbf{Z})$!

Somehow, we could say that G sits inside $\mathrm{PL}_+(\mathbf{S}^1)$ like a fuchsian group sits inside $\mathrm{PSL}(2, \mathbf{R})$. For more information concerning this group, see [13, 28].