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This bijection is an anti-homomorphism: $\overline{\sigma_1\sigma_2} = \overline{\sigma_2}\overline{\sigma_1}$. This bijection induces an operation on $\mathbf{Z}^{(k)}_{(3)}$.

THEOREM 2.9. *The operation on $\mathbf{Z}^{(k)}_{(3)}$ induced by reversing the orientation of each component of a string link is to change each $\mu(rst)$ to $-\mu(rst)$ followed by the translation operation*

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs}l_{rt} + l_{rs}l_{st} - l_{rt}l_{st}.$$

Proof. Consider the normal form (2) of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ in the r, s, t -th components. The normal form for $\overline{\sigma}$ is obtained as follows:

$$\begin{aligned} \overline{\sigma} &= [\tau_{rt}, \tau_{st}]^{-\delta} \tau_{st}^{\gamma} \tau_{rt}^{\beta} \tau_{rs}^{\alpha} \\ &= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{-\delta - \alpha\beta + \alpha\gamma - \beta\gamma}. \end{aligned}$$

Thus the operation on $\mathbf{Z}^{(k)}_{(3)}$ induced by $\sigma \mapsto \overline{\sigma}$ is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs}l_{rt} + l_{rs}l_{st} - l_{rt}l_{st}. \quad \square$$

3. CONSTRUCTION OF THE INVARIANT

By Theorems 2.2 and 2.7, we shall look for polynomials in l_{ij} and $\mu(rst)$ invariant under the translation operations on $\{\mu(rst)\} \in \mathbf{Z}^{(k)}_{(3)}$ induced by partial conjugations. There are $k(k-1)$ partial conjugations altogether and their induced translations subject to $2k$ linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of $\{l_{ij}\}$, the sublattice of $\mathbf{Z}^{(k)}_{(3)}$ generated by the translation vectors of the partial conjugations will be of dimension no larger than $k(k-1) - 2k = k^2 - 3k$.

LEMMA 3.1. *For $k > 3$, the $2k$ equations in Theorem 2.8 are linearly independent for generic values of $\{l_{ij}\}$.*

Proof. We write the two sets of equations in Theorem 2.8 as follows:

$$\begin{aligned} \mathbf{1}^i + \mathbf{2}^i + \cdots + \mathbf{j}^i + \cdots + \mathbf{k}^i &= 0, \quad j \neq i; \\ l_{i1}\mathbf{i}^1 + l_{i2}\mathbf{i}^2 + \cdots + l_{ij}\mathbf{i}^j + \cdots + l_{ik}\mathbf{i}^k &= 0, \quad j \neq i, \end{aligned}$$

for each $i = 1, 2, \dots, k$.

For generic values of $\{l_{ij}\}$, using the first $k-1$ equations from the first set of k equations, we can solve for $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^{k-1}$. Similarly, we can solve

for $\mathbf{1}^k, \mathbf{2}^k, \dots, (\mathbf{k} - \mathbf{1})^k$ from the first $k - 1$ equations of the second set of k equations. The remaining vectors \mathbf{i}^j , $i, j \neq k$, have to satisfy another two equations obtained from the last equations in those two sets of k equations, respectively, by substituting \mathbf{k}^i and \mathbf{i}^k with their solutions in terms of \mathbf{i}^j for $i, j \neq k$. It is then easy to check that these two equations are linearly independent when $k > 3$. \square

LEMMA 3.2. For $k = 4, 5$, we have $\binom{k}{3} = k^2 - 3k$. For $k \geq 6$, we have $\binom{k}{3} > k^2 - 3k$.

Proof. We have

$$\binom{k}{3} - (k^2 - 3k) = \frac{k}{6}(k^2 - 9k + 20) = \frac{k}{6}(k - 4)(k - 5). \quad \square$$

THEOREM 3.3. For $k \geq 6$, there exists a polynomial in l_{ij} and $\mu(rst)$ which is a link-homotopy invariant of ordered, oriented links with k components. This link-homotopy invariant is of finite type.

Proof. In $\mathbf{Z}^{\binom{k}{3}}$, let \mathcal{P} be the sublattice generated by the translation vectors of partial conjugations. Then we have

$$\dim(\mathcal{P}) \leq k^2 - 3k < \binom{k}{3}.$$

Let $\Omega \in \mathbf{Z}^{\binom{k}{3}}$ be a non-zero vector perpendicular to \mathcal{P} . We can choose such an Ω so that its coordinates are polynomials in $\{l_{ij}\}$ and the inner product $\mathbf{i}^j \cdot \Omega$ is identically zero. This can be achieved by considering generic values of $\{l_{ij}\}$ first and solving a system of homogeneous equations (with more equations than unknowns) whose coefficients are polynomials in l_{ij} ³). Then since $\mathbf{i}^j \cdot \Omega = 0$ for generic values of $\{l_{ij}\}$, it has to be zero identically. Let $\mu = \{\mu(rst)\} \in \mathbf{Z}^{\binom{k}{3}}$. The inner product $\mu \cdot \Omega$ is invariant under the translations by vectors in \mathcal{P} . This is a desired link-homotopy invariant of ordered, oriented links since

$$(\mu + \mathbf{i}^j) \cdot \Omega = \mu \cdot \Omega$$

for all $i, j = 1, 2, \dots, k$.

The fact that the invariant $\mu \cdot \Omega$ is of finite type is a direct consequence of the fact that the linking numbers and the triple linking numbers are all finite

³) This will be made explicit in the example following this proof.

type invariants of string links ([7], [2]). If we have a singular link, we may put it into the form of the closure of a single string link. Since polynomials of finite type invariants are still of finite type, $\mu \cdot \Omega$ vanishes on singular string links with a sufficiently large number of double points. This implies that it is a finite type link invariant. \square

We now consider in some detail the case $k = 6$. Let us order $\mu(rst)$, $1 \leq r < s < t \leq 6$ in lexicographic order. So

$$\mu = (\mu(123), \mu(124), \mu(125), \mu(126), \mu(134), \mu(135), \mu(136), \mu(145), \mu(146), \mu(156), \\ \mu(234), \mu(235), \mu(236), \mu(245), \mu(246), \mu(256), \mu(345), \mu(346), \mu(356), \mu(456)).$$

Then the vectors of the translation operations $\mathbf{1}^2, \mathbf{1}^3, \mathbf{1}^4, \mathbf{1}^5, \mathbf{1}^6, \mathbf{2}^1, \mathbf{2}^3, \mathbf{2}^4, \mathbf{2}^5, \mathbf{2}^6, \mathbf{3}^1, \mathbf{3}^2, \mathbf{3}^4, \mathbf{3}^5, \mathbf{3}^6, \mathbf{4}^1, \mathbf{4}^2, \mathbf{4}^3, \mathbf{4}^5, \mathbf{4}^6, \mathbf{5}^1, \mathbf{5}^2, \mathbf{5}^3, \mathbf{5}^4, \mathbf{5}^6, \mathbf{6}^1, \mathbf{6}^2, \mathbf{6}^3, \mathbf{6}^4, \mathbf{6}^5$ are the row vectors of the following 30×20 matrix, from top to bottom respectively:

l_{13}	l_{14}	l_{15}	l_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-l_{12}$	0	0	0	l_{14}	l_{15}	l_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0
0	$-l_{12}$	0	0	$-l_{13}$	0	0	l_{15}	l_{16}	0	0	0	0	0	0	0	0	0	0	0
0	0	$-l_{12}$	0	0	$-l_{13}$	0	$-l_{14}$	0	l_{16}	0	0	0	0	0	0	0	0	0	0
0	0	0	$-l_{12}$	0	0	$-l_{13}$	0	$-l_{14}$	$-l_{15}$	0	0	0	0	0	0	0	0	0	0
$-l_{23}$	$-l_{24}$	$-l_{25}$	$-l_{26}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_{12}	0	0	0	0	0	0	0	0	0	0	l_{24}	l_{25}	l_{26}	0	0	0	0	0	0
0	l_{12}	0	0	0	0	0	0	0	0	0	$-l_{23}$	0	0	l_{25}	l_{26}	0	0	0	0
0	0	l_{12}	0	0	0	0	0	0	0	0	0	$-l_{23}$	0	$-l_{24}$	0	l_{26}	0	0	0
0	0	0	l_{12}	0	0	0	0	0	0	0	0	0	$-l_{23}$	0	$-l_{24}$	$-l_{25}$	0	0	0
l_{23}	0	0	0	$-l_{34}$	$-l_{35}$	$-l_{36}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$-l_{13}$	0	0	0	0	0	0	0	0	0	0	$-l_{34}$	$-l_{35}$	$-l_{36}$	0	0	0	0	0	0
0	0	0	0	l_{13}	0	0	0	0	0	0	l_{23}	0	0	0	0	0	l_{35}	l_{36}	0
0	0	0	0	0	l_{13}	0	0	0	0	0	l_{23}	0	0	0	0	0	$-l_{34}$	0	l_{36}
0	0	0	0	0	0	l_{13}	0	0	0	0	0	l_{23}	0	0	0	0	$-l_{34}$	$-l_{35}$	0
0	l_{24}	0	0	l_{34}	0	0	$-l_{45}$	$-l_{46}$	0	0	0	0	0	0	0	0	0	0	0
0	$-l_{14}$	0	0	0	0	0	0	0	0	0	l_{34}	0	0	$-l_{45}$	$-l_{46}$	0	0	0	0
0	0	0	0	$-l_{14}$	0	0	0	0	0	0	$-l_{24}$	0	0	0	0	0	$-l_{45}$	$-l_{46}$	0
0	0	0	0	0	0	0	l_{14}	0	0	0	0	0	0	l_{24}	0	0	l_{34}	0	l_{46}
0	0	0	0	0	0	0	0	l_{14}	0	0	0	0	0	l_{24}	0	0	l_{34}	0	$-l_{45}$
0	0	l_{25}	0	0	l_{35}	0	l_{45}	0	$-l_{56}$	0	0	0	0	0	0	0	0	0	0
0	0	$-l_{15}$	0	0	0	0	0	0	0	0	0	l_{35}	0	l_{45}	0	$-l_{56}$	0	0	0
0	0	0	0	0	$-l_{15}$	0	0	0	0	0	0	$-l_{25}$	0	0	0	0	l_{45}	0	$-l_{56}$
0	0	0	0	0	0	0	$-l_{15}$	0	0	0	0	0	0	$-l_{25}$	0	0	$-l_{35}$	0	$-l_{56}$
0	0	0	0	0	0	0	0	0	0	l_{15}	0	0	0	0	0	l_{25}	0	0	l_{35}
0	0	0	l_{26}	0	0	l_{36}	0	l_{46}	l_{56}	0	0	0	0	0	0	0	0	0	0
0	0	0	$-l_{16}$	0	0	0	0	0	0	0	0	l_{36}	0	l_{46}	l_{56}	0	0	0	0
0	0	0	0	0	0	$-l_{16}$	0	0	0	0	0	$-l_{26}$	0	0	0	0	l_{46}	l_{56}	0
0	0	0	0	0	0	0	0	$-l_{16}$	0	0	0	0	0	$-l_{26}$	0	0	$-l_{36}$	0	l_{56}
0	0	0	0	0	0	0	0	0	$-l_{16}$	0	0	0	0	0	0	l_{26}	0	0	$-l_{36}$

We shall pick out the 18 rows of this matrix corresponding to the translation operations of $1^2, 1^3, 1^4, 1^5, 2^1, 2^3, 2^4, 2^5, 3^1, 3^2, 3^4, 3^5, 4^1, 4^2, 4^3, 4^5, 5^1, 5^2$, respectively. Calculation using *Mathematica*[®] shows that these 18 vectors are linearly independent generically.

Consider now the operation of reversing the orientation. The vector $R = \{R(rst)\} \in \mathbf{Z}^{20}$ of the translation operation in Theorem 2.9 is given by

$$R(rst) = -l_{rs} l_{rt} + l_{rs} l_{st} - l_{rt} l_{st}.$$

One can verify that the vector R and the previous 18 vectors are linearly independent. Let \mathcal{M} be the 19×20 matrix formed by these 19 vectors. Let $\mathcal{M}^{(i)}$ be the 19×19 matrix obtained from \mathcal{M} by deleting the i^{th} column from \mathcal{M} , $i = 1, 2, \dots, 20$. Let

$$\Omega_i = (-1)^{i-1} \det(\mathcal{M}^{(i)})$$

and $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_{20})$.

THEOREM 3.4. $\mu \cdot \Omega$ is a finite type link-homotopy invariant of ordered, oriented links with 6 components. When the orientation of every component is reversed, this invariant is changed only by a sign.

Proof. Using the fact that the rows of the cofactor matrix A^* of a given matrix A are perpendicular to different rows of A , we see that Ω is perpendicular to all the vectors of translation operation induced by partial conjugations as well as the vector R . Certainly, $\Omega \neq 0$. So $\mu \cdot \Omega$ is a non-trivial link-homotopy invariant of ordered, oriented links with 6 components. It is of finite type since it is a polynomial in l_{ij} and $\mu(rst)$. Under the reversion of orientation, μ changes to $-\mu + R$. Since $R \cdot \Omega = 0$, the invariant $\mu \cdot \Omega$ is only changed by a sign under the reversion of orientation. \square

To finish, let us furnish some data obtained using *Mathematica*. Let $\deg(l_{ij}) = 1$, then Ω_i is a homogeneous polynomial of degree 20 in l_{ij} . Let L_i be the number of monomials in Ω_i , the sequence $\{L_1, L_2, \dots, L_{20}\}$ is given as follows:

$$\{5531, 5555, 5555, 5531, 5424, 5769, 5802, 5734, 5753, 5432, \\ 5432, 5753, 5802, 5734, 5769, 5424, 5928, 5922, 5922, 5928\}.$$

Thus $\mu \cdot \Omega$ is linear and homogeneous in $\mu(rst)$ and has 113,700 monomials.