Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FINITE TYPE LINK-HOMOTOPY INVARIANTS
Autor: Lin, Xiao-Song
Kapitel: 3. Construction of the invariant
DOI: https://doi.org/10.5169/seals-65440

This bijection is an anti-homomorphism: $\overline{\sigma_1\sigma_2} = \overline{\sigma}_2\,\overline{\sigma}_1$. This bijection induces an operation on $\mathbf{Z}^{\binom{k}{3}}$.

THEOREM 2.9. *The operation on $\mathbf{Z}^{\binom{k}{3}}$ induced by reversing the orientation of each component of a string link is to change each $\mu(rst)$ to $-\mu(rst)$ followed by the translation operation*

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs}\,l_{rt} + l_{rs}\,l_{st} - l_{rt}\,l_{st}\,.$$

Proof. Consider the normal form (2) of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ in the r,s,t-th components. The normal form for $\overline{\sigma}$ is obtained as follows:

$$\begin{aligned}\overline{\sigma} &= [\tau_{rt},\tau_{st}]^{-\delta}\tau_{st}^{\gamma}\tau_{rt}^{\beta}\tau_{rs}^{\alpha} \\ &= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{-\delta-\alpha\beta+\alpha\gamma-\beta\gamma}\,.\end{aligned}$$

Thus the operation on $Z^{\binom{k}{3}}$ induced by $\sigma \mapsto \bar{\sigma}$ is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs}\,l_{rt} + l_{rs}\,l_{st} - l_{rt}\,l_{st}\,. \qquad \square$$

3. CONSTRUCTION OF THE INVARIANT

By Theorems 2.2 and 2.7, we shall look for polynomials in l_{ij} and $\mu(rst)$ invariant under the translation operations on $\{\mu(rst)\} \in \mathbf{Z}^{\binom{k}{3}}$ induced by partial conjugations. There are $k(k-1)$ partial conjugations altogether and their induced translations subject to $2k$ linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of $\{l_{ij}\}$, the sublattice of $Z^{\binom{k}{3}}$ generated by the translation vectors of the partial conjugations will be of dimension no larger than $k(k-1)-2k = k^2-3k$.

LEMMA 3.1. *For $k>3$, the $2k$ equations in Theorem 2.8 are linearly independent for generic values of $\{l_{ij}\}$.*

Proof. We write the two sets of equations in Theorem 2.8 as follows:

$$\mathbf{1}^i + \mathbf{2}^i + \cdots + \mathbf{j}^i + \cdots + \mathbf{k}^i = 0, \quad j \neq i;$$

$$l_{i1}\mathbf{i}^1 + l_{i2}\mathbf{i}^2 + \cdots + l_{ij}\mathbf{i}^j + \cdots + l_{ik}\mathbf{i}^k = 0, \quad j \neq i,$$

for each $i = 1,2,\ldots,k$.

For generic values of $\{l_{ij}\}$, using the first $k-1$ equations from the first set of k equations, we can solve for $\mathbf{k}^1,\mathbf{k}^2,\ldots,\mathbf{k}^{k-1}$. Similarly, we can solve for $\mathbf{1}^k, \mathbf{2}^k, \ldots, (\mathbf{k}-\mathbf{1})^k$ from the first $k-1$ equations of the second set of k equations. The remaining vectors $\mathbf{i}^j$, $i,j \neq k$, have to satisfy another two equations obtained from the last equations in those two sets of k equations, respectively, by substituting $\mathbf{k}^i$ and $\mathbf{i}^k$ with their solutions in terms of $\mathbf{i}^j$ for $i,j \neq k$. It it then easy to check that these two equations are linearly independent when $k > 3$. □

LEMMA 3.2. *For $k = 4,5$, we have $\binom{k}{3} = k^2 - 3k$. For $k \geq 6$, we have $\binom{k}{3} > k^2 - 3k$.*

Proof. We have

$$\binom{k}{3} - (k^2 - 3k) = \frac{k}{6}(k^2 - 9k + 20) = \frac{k}{6}(k-4)(k-5). \qquad \square$$

THEOREM 3.3. *For $k \geq 6$, there exists a polynomial in l_{ij} and $\mu(rst)$ which is a link-homotopy invariant of ordered, oriented links with k components. This link-homotopy invariant is of finite type.*

Proof. In $\mathbf{Z}^{\binom{k}{3}}$, let $\mathcal{P}$ be the sublattice generated by the translation vectors of partial conjugations. Then we have

$$\dim(\mathcal{P}) \leq k^2 - 3k < \binom{k}{3}.$$

Let $\Omega \in \mathbf{Z}^{\binom{k}{3}}$ be a non-zero vector perpendicular to $\mathcal{P}$. We can choose such an Ω so that its coordinates are polynomials in $\{l_{ij}\}$ and the inner product $\mathbf{i}^j \cdot \Omega$ is identically zero. This can be achieved by considering generic values of $\{l_{ij}\}$ first and solving a system of homogeneous equations (with more equations than unknowns) whose coefficients are polynomials in l_{ij} [3]). Then since $\mathbf{i}^j \cdot \Omega = 0$ for generic values of $\{l_{ij}\}$, it has to be zero identically. Let $\mu = \{\mu(rst)\} \in \mathbf{Z}^{\binom{k}{3}}$. The inner product $\mu \cdot \Omega$ is invariant under the translations by vectors in $\mathcal{P}$. This is a desired link-homotopy invariant of ordered, oriented links since

$$(\mu + \mathbf{i}^j) \cdot \Omega = \mu \cdot \Omega$$

for all $i,j = 1,2,\ldots,k$.

The fact that the invariant $\mu \cdot \Omega$ is of finite type is a direct consequence of the fact that the linking numbers and the triple linking numbers are all finite

[3]) This will be made explicit in the example following this proof.

type invariants of string links ([7], [2]). If we have a singular link, we may put it into the form of the closure of a single string link. Since polynomials of finite type invariants are still of finite type, $\mu \cdot \Omega$ vanishes on singular string links with a sufficiently large number of double points. This implies that it is a finite type link invariant. □

We now consider in some detail the case $k = 6$. Let us order $\mu(rst)$, $1 \leq r < s < t \leq 6$ in lexicographic order. So

$$\mu = (\mu(123), \mu(124), \mu(125), \mu(126), \mu(134), \mu(135), \mu(136), \mu(145), \mu(146), \mu(156), \mu(234), \mu(235), \mu(236), \mu(245), \mu(246), \mu(256), \mu(345), \mu(346), \mu(356), \mu(456)).$$

Then the vectors of the translation operations $\mathbf{1}^2$, $\mathbf{1}^3$, $\mathbf{1}^4$, $\mathbf{1}^5$, $\mathbf{1}^6$, $\mathbf{2}^1$, $\mathbf{2}^3$, $\mathbf{2}^4$, $\mathbf{2}^5$, $\mathbf{2}^6$, $\mathbf{3}^1$, $\mathbf{3}^2$, $\mathbf{3}^4$, $\mathbf{3}^5$, $\mathbf{3}^6$, $\mathbf{4}^1$, $\mathbf{4}^2$, $\mathbf{4}^3$, $\mathbf{4}^5$, $\mathbf{4}^6$, $\mathbf{5}^1$, $\mathbf{5}^2$, $\mathbf{5}^3$, $\mathbf{5}^4$, $\mathbf{5}^6$, $\mathbf{6}^1$, $\mathbf{6}^2$, $\mathbf{6}^3$, $\mathbf{6}^4$, $\mathbf{6}^5$ are the row vectors of the following 30×20 matrix, from top to bottom respectively:

$$\left|\begin{array}{cccccccccccccccccccc}
l_{13} & l_{14} & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{12} & 0 & 0 & 0 & l_{14} & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -l_{12} & 0 & 0 & -l_{13} & 0 & 0 & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -l_{12} & 0 & 0 & -l_{13} & 0 & -l_{14} & 0 & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -l_{12} & 0 & 0 & -l_{13} & 0 & -l_{14} & -l_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{23} & -l_{24} & -l_{25} & -l_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{24} & l_{25} & l_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & 0 & l_{25} & l_{26} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & -l_{24} & 0 & l_{26} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & -l_{24} & -l_{25} & 0 & 0 & 0 & 0 \\
l_{23} & 0 & 0 & 0 & -l_{34} & -l_{35} & -l_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{34} & -l_{35} & -l_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & 0 & l_{35} & l_{36} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & -l_{34} & 0 & l_{36} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & -l_{34} & -l_{35} & 0 \\
0 & l_{24} & 0 & 0 & l_{34} & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -l_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{34} & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -l_{14} & 0 & 0 & 0 & 0 & 0 & -l_{24} & 0 & 0 & 0 & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{14} & 0 & 0 & 0 & 0 & 0 & l_{24} & 0 & 0 & l_{34} & 0 & 0 & l_{46} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{14} & 0 & 0 & 0 & 0 & 0 & l_{24} & 0 & 0 & l_{34} & 0 & -l_{45} \\
0 & 0 & l_{25} & 0 & 0 & l_{35} & 0 & l_{45} & 0 & -l_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{35} & 0 & l_{45} & 0 & -l_{56} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & -l_{25} & 0 & 0 & 0 & 0 & l_{45} & 0 & -l_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & -l_{25} & 0 & 0 & -l_{35} & 0 & 0 & -l_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{15} & 0 & 0 & 0 & 0 & 0 & l_{25} & 0 & 0 & l_{35} & l_{45} \\
0 & 0 & 0 & l_{26} & 0 & 0 & l_{36} & 0 & l_{46} & l_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{36} & 0 & l_{46} & l_{56} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & -l_{26} & 0 & 0 & 0 & 0 & l_{46} & l_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & -l_{26} & 0 & 0 & -l_{36} & 0 & l_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & l_{26} & 0 & 0 & -l_{36} & -l_{46}
\end{array}\right|$$

We shall pick out the 18 rows of this matrix corresponding to the translation operations of $\mathbf{1}^2$, $\mathbf{1}^3$, $\mathbf{1}^4$, $\mathbf{1}^5$, $\mathbf{2}^1$, $\mathbf{2}^3$, $\mathbf{2}^4$, $\mathbf{2}^5$, $\mathbf{3}^1$, $\mathbf{3}^2$, $\mathbf{3}^4$, $\mathbf{3}^5$, $\mathbf{4}^1$, $\mathbf{4}^2$, $\mathbf{4}^3$, $\mathbf{4}^5$, $\mathbf{5}^1$, $\mathbf{5}^2$, respectively. Calculation using *Mathematica*® shows that these 18 vectors are linearly independent generically.

Consider now the operation of reversing the orientation. The vector $R = \{R(rst)\} \in \mathbf{Z}^{20}$ of the translation operation in Theorem 2.9 is given by

$$R(rst) = -l_{rs}\, l_{rt} + l_{rs}\, l_{st} - l_{rt}\, l_{st}\,.$$

One can verify that the vector R and the previous 18 vectors are linearly independent. Let $\mathcal{M}$ be the 19×20 matrix formed by these 19 vectors. Let $\mathcal{M}^{(i)}$ be the 19×19 matrix obtained from $\mathcal{M}$ by deleting the i^{th} column from $\mathcal{M}$, $i = 1, 2, \ldots, 20$. Let

$$\Omega_i = (-1)^{i-1} \det(\mathcal{M}^{(i)})$$

and $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{20})$.

THEOREM 3.4. *$\mu \cdot \Omega$ is a finite type link-homotopy invariant of ordered, oriented links with 6 components. When the orientation of every component is reversed, this invariant is changed only by a sign.*

Proof. Using the fact that the rows of the cofactor matrix A^* of a given matrix A are perpendicular to different rows of A, we see that Ω is perpendicular to all the vectors of translation operation induced by partial conjugations as well as the vector R. Certainly, $\Omega \neq 0$. So $\mu \cdot \Omega$ is a non-trivial link-homotopy invariant of ordered, oriented links with 6 components. It is of finite type since it is a polynomial in l_{ij} and $\mu(rst)$. Under the reversion of orientation, μ changes to $-\mu + R$. Since $R \cdot \Omega = 0$, the invariant $\mu \cdot \Omega$ is only changed by a sign under the reversion of orientation. □

To finish, let us furnish some data obtained using *Mathematica.* Let $\deg(l_{ij}) = 1$, then Ω_i is a homogeneous polynomial of degree 20 in l_{ij}. Let L_i be the number of monomials in Ω_i, the sequence $\{L_1, L_2, \ldots, L_{20}\}$ is given as follows:

$$\{5531, 5555, 5555, 5531, 5424, 5769, 5802, 5734, 5753, 5432,$$
$$5432, 5753, 5802, 5734, 5769, 5424, 5928, 5922, 5922, 5928\}\,.$$

Thus $\mu \cdot \Omega$ is linear and homogeneous in $\mu(rst)$ and has 113,700 monomials.