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well as the work of Mellor and Thurston, of course) shows the existence of non-trivial finite type link concordance invariants.

To extend the applicability of our general philosophy slightly, we find that the operation on the vector  $\{\mu(rst)\}$  induced by reversing the orientation of each component of a string link is to change it by a negative sign followed by a translation whose translation vector's coordinates are quadratic polynomials in  $l_{ii}$ . If the dimension of the subspace generated by this vector together with the translation vectors of conjugations and partial conjugations is still less than  $\binom{k}{3}$ for generic values of the linking numbers, and this is the case indeed, we can construct a non-trivial link-homotopy invariant polynomial which is changed by a sign when the orientation of each component of a link is reversed. We say that such a link invariant detects the invertibility for links. Recall that the reversion of the orientation of every component of a link does not change the quantum invariant associated with an irreducible representation of a semisimple Lie algebra (see, for example, [8]). Thus our invariant is of finite type but is not determined by quantum invariants. The existence of a finite type knot invariant which detects the invertibility for knots is a major problem in the theory of finite type invariants (see, for example, [8] and [4]). We believe that finite type knot invariants can not detect the invertibility for knots.

It remains unclear whether we can have a complete set of link-homotopy invariant polynomials which determines uniquely link-homotopy classes of links. See [5] for an earlier attempt on this problem<sup>2</sup>). This problem could probably be translated to the problem of understanding the sublattice generated by the translation vectors of conjugations and partial conjugations. A better understanding of this sublattice might also be useful in answering the following question. If we let  $\deg(l_{ij}) = 1$  and  $\deg(\mu(rst)) = 2$ , the link-homotopy invariant polynomial for k = 6 we construct in Section 3, which detects the invertibility for links, is a linear combination of 113,700 monomials of degree 22, homogeneous in both  $l_{ij}$  and  $\mu(rst)$  and linear in  $\mu(rst)$ . Is there a shorter link-homotopy invariant polynomial detecting the invertibility for links?

## 2. CONJUGATION AND PARTIAL CONJUGATION

We first recall the classification of ordered, oriented links up to link-homotopy given in [3]. We will follow the notations of [3].

<sup>&</sup>lt;sup>2</sup>) See [6] for another approach to the similar problem for surgery equivalence of links. Notice that both approaches attempted to reduce the indeterminacies of the  $\bar{\mu}$ -invariants.

Let  $\mathcal{H}(k)$  be the group of link-homotopy classes of ordered, oriented string links with k components. The components of a string link will be ordered by  $1, 2, \ldots, k$ . Recall that a string link is a concordance of k marked points inside of the 2-disk  $D^2$  to itself in  $D^2 \times [0, 1]$ , such that it has no closed component. Two string links are link-homotopic if they are homotopic in such a way that at any moment of the homotopy, different components remain disjoint (but they are allowed to have self-intersections). Two string links can be put together to form a new string link and this gives rise to a group structure on the set of all link-homotopy classes of string links. This is the group  $\mathcal{H}(k)$ .

A pure braid is by definition a string link of the same number of components. So we have a natural map from the pure braid group P(k) of k components to  $\mathcal{H}(k)$ . It is shown in [3] that this natural map  $P(k) \to \mathcal{H}(k)$  is onto.

Deletion of the  $i^{th}$  component of the string link gives rise to a group homomorphism  $d_i \colon \mathcal{H}(k) \to \mathcal{H}(k-1)$ . If F(k) denotes the free group of rank k generated by  $x_1, x_2, \ldots, x_k$ , the reduced free group RF(k) is the quotient of F(k) by adding relations  $[x_i, x_i^g] = 1$  for all i and all  $g \in F(k)$ .

LEMMA 2.1. There is a split short exact sequence of groups

(1) 
$$1 \longrightarrow RF(k-1) \longrightarrow \mathcal{H}(k) \xrightarrow{d_i} \mathcal{H}(k-1) \longrightarrow 1$$

where RF(k-1) is the reduced free group generated by  $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$ .

Notice that the split exact sequence (1) depends on the deleted component so that there are k such split exact sequences altogether. A split exact sequence determines a semi-direct product decomposition

$$\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1).$$

Conjugation in the group  $\mathcal{H}(k)$  is defined as usual: A conjugation of  $\sigma \in \mathcal{H}(k)$  by  $\beta \in \mathcal{H}(k)$  is the element  $\beta \sigma \beta^{-1} \in \mathcal{H}(k)$ . A partial conjugation of  $\sigma \in \mathcal{H}(k)$  is an element of the form  $\theta hgh^{-1}$ , where we write  $\sigma = \theta g$  according to a decomposition  $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$ , for  $\theta \in \mathcal{H}(k-1)$  and  $g \in RF(k-1)$ , and for an arbitrary  $h \in RF(k-1)$ .

To form the closure of a string link  $\sigma \in \mathcal{H}(k)$ , we may think of it as a pure braid in P(k) and its closure will be the usual braid closure. The closure of  $\sigma \in \mathcal{H}(k)$  is an ordered, oriented link of k components. It is not hard to see that every link-homotopy class of ordered, oriented links with k components can be realized as the closure of an element in  $\mathcal{H}(k)$ , and thus the closure of a pure braid in P(k). One of the main results of [3] is the following classification theorem.

THEOREM 2.2. Let  $\sigma, \sigma' \in \mathcal{H}(k)$ . Then the closures of  $\sigma$  and  $\sigma'$  are link-homotopic as ordered, oriented links if and only if there is a sequence  $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$  of elements of  $\mathcal{H}(k)$  such that  $\sigma_{j+1}$  is either a conjugation or a partial conjugation of  $\sigma_j$ .

For a group G, we will denote by  $G_n$  the  $n^{\text{th}}$  term of the lower central series of G, i.e.  $G_1 = G$  and  $G_{n+1} = [G_n, G]$ , the normal subgroup of G generated by elements of the form  $[g,h] = ghg^{-1}h^{-1}$  for all  $g \in G_n$  and  $h \in G$ . A group G is nilpotent of class n if  $G_{n+1} = 1$  but  $G_n \neq 1$ . We summarize some known facts about the group structures of  $\mathcal{H}(k)$  in the following lemma.

LEMMA 2.3. 1)  $\mathcal{H}(k)$  is torsion free and nilpotent of class k-1.

2) Corresponding to a decomposition  $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$ , we have

$$H(k)_n = \mathcal{H}(k-1)_n \ltimes RF(k-1)_n.$$

3)  $\mathcal{H}(k)_{n-1}/\mathcal{H}(k)_n$  is a free abelian group of rank  $(n-2)!\binom{k}{n}$ .

For  $\sigma \in \mathcal{H}(k)$ , its image in  $\mathcal{H}(k)/\mathcal{H}(k)_3$  can be described by  $\binom{k}{2} + \binom{k}{3}$  integers. These integers are linking numbers  $l_{ij}$ , for  $1 \leq i < j \leq k$ , and Milnor's triple linking numbers  $\mu(rst)$ , for  $1 \leq r < s < t \leq k$ . We want to have them defined precisely and understand how they change when  $\sigma$  is changed by a conjugation or a partial conjugation.

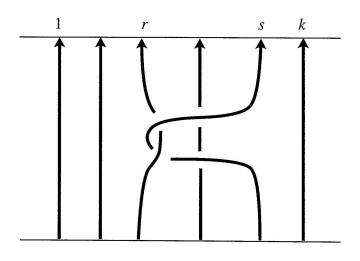


FIGURE 1
The pure braid  $au_{rs}$ 

We will denote by  $\tau_{rs} = \tau_{sr}$ , for  $1 \le r < s \le k$ , the pure braid depicted in Figure 1. Let  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ . For  $1 \le r < s < t \le k$ , after deleting

all components other than the r, s, t-th components,  $\sigma$  can be written in the following normal form

(2) 
$$\sigma = \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta},$$

where  $\alpha = l_{rs}$ ,  $\beta = l_{rt}$ ,  $\gamma = l_{st}$ . By definition, we have  $\delta = \mu(rst)$  for  $\sigma \in \mathcal{H}(k)$ .

LEMMA 2.4. In  $\mathcal{H}(k)/\mathcal{H}(k)_3$ , if r', s', t' is a permutation of r, s, t and  $\epsilon$  is the sign of the permutation, then

$$[\tau_{r't'},\tau_{s't'}]=[\tau_{rt},\tau_{st}]^{\epsilon}.$$

Furthermore, we have

$$[\tau_{rt}^{\eta},\tau_{st}]=[\tau_{rt},\tau_{st}]^{\eta}.$$

This lemma is useful in the following calculation and its proof is straightforward.

To understand how  $\mu(rst)$  changes under the conjugation, we only need to calculate the conjugation of  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  under the normal form (2) by  $\tau_{rs}, \tau_{rt}, \tau_{st}$ . This calculation is straightforward:

$$\tau_{rs}\sigma\tau_{rs}^{-1} = \tau_{rs}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rs}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rs},\tau_{rt}]^{\beta}[\tau_{rs},\tau_{st}]^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\beta-\gamma};$$

$$\tau_{rt}\sigma\tau_{rt}^{-1} = \tau_{rt}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rt}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{rs}]^{\alpha}[\tau_{rt},\tau_{st}]^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\alpha+\gamma};$$

$$\tau_{st}\sigma\tau_{st}^{-1} = \tau_{st}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{st}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{st},\tau_{rs}]^{\alpha}[\tau_{st},\tau_{rt}]^{\beta}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\alpha-\beta}.$$

We summarize the calculation into the following lemma.

LEMMA 2.5. The change of  $\mu(rst)$  under a conjugation is given by

Conjugation by  $\tau_{rs}$ :  $\mu(rst) \rightarrow \mu(rst) + l_{rt} - l_{st}$ ;

Conjugation by  $\tau_{rt}$ :  $\mu(rst) \rightarrow \mu(rst) - l_{rs} + l_{st}$ ;

Conjugation by  $\tau_{st}$ :  $\mu(rst) \rightarrow \mu(rst) + l_{rs} - l_{rt}$ .

Furthermore,  $\mu(rst)$  will not change under a conjugation by  $\tau_{ij}$  where  $\{i,j\}$  and  $\{r,s,t\}$  have at most one element in common.

The calculation of partial conjugations is slightly more complicated. We will start with partial conjugations by  $\tau_{rt}$  and  $\tau_{st}$ . These two operations are denoted by  $\mathbf{t}^r$  and  $\mathbf{t}^s$ , respectively. For  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  under the normal form (2), we have:

$$\sigma \xrightarrow{\mathbf{t}^{r}} \tau_{rs}^{\alpha} \tau_{rt} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta} \tau_{rt}^{-1}$$

$$= \tau_{rs} \alpha \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta + \gamma};$$

$$\sigma \xrightarrow{\mathbf{t}^{s}} \tau_{rs}^{\alpha} \tau_{st} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta} \tau_{st}^{-1}$$

$$= \tau_{rs} \alpha \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta - \beta}.$$

To calculate partial conjugations by  $\tau_{rs}$  and  $\tau_{ts}$ , which are denoted by  $\mathbf{s}^r$  and  $\mathbf{s}^t$ , respectively, we need to rewrite  $\sigma$  as follows:

$$\sigma = \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta} = \tau_{rt}^{\beta} \tau_{rs}^{\alpha} \tau_{ts}^{\gamma} [\tau_{rs}, \tau_{ts}]^{-\delta - \alpha\beta}.$$

Then, we have:

$$\sigma \xrightarrow{\mathbf{s}^{r}} \tau_{rs}^{\beta} \tau_{rs} \tau_{rs}^{\alpha} \tau_{rs}^{\gamma} [\tau_{rs}, \tau_{ts}]^{-\delta - \alpha \beta} \tau_{rs}^{-1}$$

$$= \tau_{rt}^{\beta} \tau_{rs}^{\alpha} \tau_{ts}^{\gamma} [\tau_{rs}, \tau_{ts}]^{-\delta - \alpha \beta + \gamma}$$

$$= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta - \gamma};$$

$$\sigma \xrightarrow{\mathbf{s}^{t}} \tau_{rt}^{\beta} \tau_{ts} \tau_{rs}^{\alpha} \tau_{ts}^{\gamma} [\tau_{rs}, \tau_{ts}]^{-\delta - \alpha \beta} \tau_{ts}^{-1}$$

$$= \tau_{rt}^{\beta} \tau_{rs}^{\alpha} \tau_{ts}^{\gamma} [\tau_{rs}, \tau_{ts}]^{-\delta - \alpha \beta - \alpha}$$

$$= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta + \gamma}.$$

Similarly, to calculate partial conjugations  $\mathbf{r}^s$  and  $\mathbf{r}^t$ , we first rewrite  $\sigma$ :

$$\sigma = \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta} = \tau_{st}^{\gamma} \tau_{sr}^{\alpha} \tau_{tr}^{\beta} [\tau_{sr}, \tau_{tr}]^{\delta - \alpha \gamma + \beta \gamma}.$$

Then, we have

$$\sigma \xrightarrow{\mathbf{r}^{s}} \tau_{st}^{\gamma} \tau_{sr} \tau_{sr}^{\alpha} \tau_{tr}^{\beta} [\tau_{sr}, \tau_{tr}]^{\delta - \alpha \gamma + \beta \gamma} \tau_{sr}^{-1}$$

$$= \tau_{st}^{\gamma} \tau_{sr}^{\alpha} \tau_{tr}^{\beta} [\tau_{sr}, \tau_{tr}]^{\delta - \alpha \gamma + \beta \gamma + \beta}$$

$$= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta + \beta};$$

$$\sigma \xrightarrow{\mathbf{r}^{t}} \tau_{st}^{\gamma} \tau_{tr} \tau_{sr}^{\alpha} \tau_{tr}^{\beta} [\tau_{sr}, \tau_{tr}]^{\delta - \alpha \gamma + \beta \gamma} \tau_{tr}^{-1}$$

$$= \tau_{st}^{\gamma} \tau_{sr}^{\alpha} \tau_{tr}^{\beta} [\tau_{sr}, \tau_{tr}]^{\delta - \alpha \gamma + \beta \gamma - \alpha}$$

$$= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{\delta - \alpha}.$$

We summarize the previous calculation into the following lemma.

LEMMA 2.6. The change of  $\mu(rst)$  under a partial conjugation is given by

$$\mathbf{t}^{r}$$
:  $\mu(rst) \rightarrow \mu(rst) + l_{st}$ ;  
 $\mathbf{t}^{s}$ :  $\mu(rst) \rightarrow \mu(rst) - l_{rt}$ ;  
 $\mathbf{s}^{r}$ :  $\mu(rst) \rightarrow \mu(rst) - l_{st}$ ;  
 $\mathbf{s}^{t}$ :  $\mu(rst) \rightarrow \mu(rst) + l_{rs}$ ;  
 $\mathbf{r}^{s}$ :  $\mu(rst) \rightarrow \mu(rst) + l_{rt}$ ;  
 $\mathbf{r}^{t}$ :  $\mu(rst) \rightarrow \mu(rst) - l_{rs}$ .

Furthermore, a partial conjugation by  $\mathbf{i}^j$  will not change  $\mu(rst)$  if  $\{i,j\}$  and  $\{r,s,t\}$  have at most one element in common.

For a given string link  $\sigma \in \mathcal{H}(k)$ , we will think of the whole collection  $\{\mu(rst); 1 \leq r < s < t \leq k\}$  as an element in  $\mathbf{Z}^{\binom{k}{3}}$ . Then the conjugations and partial conjugations act on  $\mathbf{Z}^{\binom{k}{3}}$  by translations. We will abuse the notation by using the same symbol to denote both a translation operation and the corresponding translation vector. Thus, a translation operation  $T: V \to V$  on a vector space V is given by T(v) = v + T, for all  $v \in V$  and a fixed  $T \in V$ . If  $T_1$  and  $T_2$  are two translations, we have

$$(T_1 \cdot T_2)(v) = v + T_1 + T_2,$$
 for all  $v \in V$ .

The following two theorems follow directly from Lemmas 2.5 and 2.6.

THEOREM 2.7. The translation operation on  $\mathbf{Z}^{\binom{k}{3}}$  given by the conjugation of  $\tau_{ij}$  is the same as the composition of the translation operations given by the partial conjugations  $\mathbf{i}^j$  and  $\mathbf{j}^i$ , i.e. it is equal to  $\mathbf{i}^j + \mathbf{j}^i$ .

Theorem 2.8. The translation operations  $\mathbf{i}^j$  satisfy the following relations:

$$\sum_{j\neq i}\mathbf{j}^i=0\,,\qquad \sum_{j\neq i}l_{ij}\,\mathbf{i}^j=0$$

for all i = 1, 2, ..., k.

String links are oriented in the sense that each component is given an orientation from the bottom to the top. See Figure 1. Reversing the orientation on each component of a string link defines a bijection

$$\sigma \mapsto \overline{\sigma} \colon \mathcal{H}(k) \to \mathcal{H}(k)$$
.

This bijection is an anti-homomorphism:  $\overline{\sigma_1 \sigma_2} = \overline{\sigma_2} \, \overline{\sigma_1}$ . This bijection induces an operation on  $\mathbf{Z}^{\binom{k}{3}}$ .

THEOREM 2.9. The operation on  $\mathbf{Z}^{\binom{k}{3}}$  induced by reversing the orientation of each component of a string link is to change each  $\mu(rst)$  to  $-\mu(rst)$  followed by the translation operation

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs} l_{rt} + l_{rs} l_{st} - l_{rt} l_{st}$$
.

*Proof.* Consider the normal form (2) of  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  in the r, s, t-th components. The normal form for  $\overline{\sigma}$  is obtained as follows:

$$\overline{\sigma} = [\tau_{rt}, \tau_{st}]^{-\delta} \tau_{st}^{\gamma} \tau_{rt}^{\beta} \tau_{rs}^{\alpha} 
= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{-\delta - \alpha\beta + \alpha\gamma - \beta\gamma}.$$

Thus the operation on  $Z^{\binom{k}{3}}$  induced by  $\sigma \mapsto \bar{\sigma}$  is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs} l_{rt} + l_{rs} l_{st} - l_{rt} l_{st}$$
.

# 3. Construction of the invariant

By Theorems 2.2 and 2.7, we shall look for polynomials in  $l_{ij}$  and  $\mu(rst)$  invariant under the translation operations on  $\{\mu(rst)\}\in \mathbf{Z}^{\binom{k}{3}}$  induced by partial conjugations. There are k(k-1) partial conjugations altogether and their induced translations subject to 2k linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of  $\{l_{ij}\}$ , the sublattice of  $Z^{\binom{k}{3}}$  generated by the translation vectors of the partial conjugations will be of dimension no larger than  $k(k-1)-2k=k^2-3k$ .

LEMMA 3.1. For k > 3, the 2k equations in Theorem 2.8 are linearly independent for generic values of  $\{l_{ij}\}$ .

*Proof.* We write the two sets of equations in Theorem 2.8 as follows:

$$\mathbf{1}^{i} + \mathbf{2}^{i} + \dots + \mathbf{j}^{i} + \dots + \mathbf{k}^{i} = 0, \quad j \neq i;$$
  
$$l_{i1}\mathbf{i}^{1} + l_{i2}\mathbf{i}^{2} + \dots + l_{ij}\mathbf{i}^{j} + \dots + l_{ik}\mathbf{i}^{k} = 0, \quad j \neq i,$$

for each i = 1, 2, ..., k.

For generic values of  $\{l_{ij}\}$ , using the first k-1 equations from the first set of k equations, we can solve for  $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^{k-1}$ . Similarly, we can solve