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well as the work of Mellor and Thurston, of course) shows the existence of non-trivial finite type link concordance invariants.

To extend the applicability of our general philosophy slightly, we find that the operation on the vector  $\{\mu(rst)\}$  induced by reversing the orientation of each component of a string link is to change it by a negative sign followed by a translation whose translation vector's coordinates are quadratic polynomials in  $l_{ij}$ . If the dimension of the subspace generated by this vector together with the translation vectors of conjugations and partial conjugations is still less than  $\binom{k}{3}$  for generic values of the linking numbers, and this is the case indeed, we can construct a non-trivial link-homotopy invariant polynomial which is changed by a sign when the orientation of each component of a link is reversed. We say that such a link invariant detects the invertibility for links. Recall that the reversion of the orientation of every component of a link does not change the quantum invariant associated with an irreducible representation of a semi-simple Lie algebra (see, for example, [8]). Thus our invariant is of finite type but is not determined by quantum invariants. The existence of a finite type knot invariant which detects the invertibility for knots is a major problem in the theory of finite type invariants (see, for example, [8] and [4]). We believe that finite type knot invariants can not detect the invertibility for knots.

It remains unclear whether we can have a complete set of link-homotopy invariant polynomials which determines uniquely link-homotopy classes of links. See [5] for an earlier attempt on this problem<sup>2</sup>). This problem could probably be translated to the problem of understanding the sublattice generated by the translation vectors of conjugations and partial conjugations. A better understanding of this sublattice might also be useful in answering the following question. If we let  $\deg(l_{ij}) = 1$  and  $\deg(\mu(rst)) = 2$ , the link-homotopy invariant polynomial for  $k = 6$  we construct in Section 3, which detects the invertibility for links, is a linear combination of 113,700 monomials of degree 22, homogeneous in both  $l_{ij}$  and  $\mu(rst)$  and linear in  $\mu(rst)$ . Is there a shorter link-homotopy invariant polynomial detecting the invertibility for links?

## 2. CONJUGATION AND PARTIAL CONJUGATION

We first recall the classification of ordered, oriented links up to link-homotopy given in [3]. We will follow the notations of [3].

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<sup>2</sup>) See [6] for another approach to the similar problem for surgery equivalence of links. Notice that both approaches attempted to reduce the indeterminacies of the  $\bar{\mu}$ -invariants.

Let  $\mathcal{H}(k)$  be the group of link-homotopy classes of ordered, oriented string links with  $k$  components. The components of a string link will be ordered by  $1, 2, \dots, k$ . Recall that a string link is a concordance of  $k$  marked points inside of the 2-disk  $D^2$  to itself in  $D^2 \times [0, 1]$ , such that it has no closed component. Two string links are link-homotopic if they are homotopic in such a way that at any moment of the homotopy, different components remain disjoint (but they are allowed to have self-intersections). Two string links can be put together to form a new string link and this gives rise to a group structure on the set of all link-homotopy classes of string links. This is the group  $\mathcal{H}(k)$ .

A pure braid is by definition a string link of the same number of components. So we have a natural map from the pure braid group  $P(k)$  of  $k$  components to  $\mathcal{H}(k)$ . It is shown in [3] that this natural map  $P(k) \rightarrow \mathcal{H}(k)$  is onto.

Deletion of the  $i^{\text{th}}$  component of the string link gives rise to a group homomorphism  $d_i: \mathcal{H}(k) \rightarrow \mathcal{H}(k-1)$ . If  $F(k)$  denotes the free group of rank  $k$  generated by  $x_1, x_2, \dots, x_k$ , the reduced free group  $RF(k)$  is the quotient of  $F(k)$  by adding relations  $[x_i, x_i^g] = 1$  for all  $i$  and all  $g \in F(k)$ .

LEMMA 2.1. *There is a split short exact sequence of groups*

$$(1) \quad 1 \longrightarrow RF(k-1) \longrightarrow \mathcal{H}(k) \xrightarrow{d_i} \mathcal{H}(k-1) \longrightarrow 1$$

where  $RF(k-1)$  is the reduced free group generated by  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k$ .

Notice that the split exact sequence (1) depends on the deleted component so that there are  $k$  such split exact sequences altogether. A split exact sequence determines a semi-direct product decomposition

$$\mathcal{H}(k) = \mathcal{H}(k-1) \rtimes RF(k-1).$$

Conjugation in the group  $\mathcal{H}(k)$  is defined as usual: A conjugation of  $\sigma \in \mathcal{H}(k)$  by  $\beta \in \mathcal{H}(k)$  is the element  $\beta\sigma\beta^{-1} \in \mathcal{H}(k)$ . A *partial conjugation* of  $\sigma \in \mathcal{H}(k)$  is an element of the form  $\theta h g h^{-1}$ , where we write  $\sigma = \theta g$  according to a decomposition  $\mathcal{H}(k) = \mathcal{H}(k-1) \rtimes RF(k-1)$ , for  $\theta \in \mathcal{H}(k-1)$  and  $g \in RF(k-1)$ , and for an arbitrary  $h \in RF(k-1)$ .

To form the closure of a string link  $\sigma \in \mathcal{H}(k)$ , we may think of it as a pure braid in  $P(k)$  and its closure will be the usual braid closure. The closure of  $\sigma \in \mathcal{H}(k)$  is an ordered, oriented link of  $k$  components. It is not hard to see that every link-homotopy class of ordered, oriented links with  $k$  components can be realized as the closure of an element in  $\mathcal{H}(k)$ , and thus the closure of a pure braid in  $P(k)$ . One of the main results of [3] is the following classification theorem.

**THEOREM 2.2.** *Let  $\sigma, \sigma' \in \mathcal{H}(k)$ . Then the closures of  $\sigma$  and  $\sigma'$  are link-homotopic as ordered, oriented links if and only if there is a sequence  $\sigma = \sigma_0, \sigma_1, \dots, \sigma_n = \sigma'$  of elements of  $\mathcal{H}(k)$  such that  $\sigma_{j+1}$  is either a conjugation or a partial conjugation of  $\sigma_j$ .*

For a group  $G$ , we will denote by  $G_n$  the  $n^{\text{th}}$  term of the lower central series of  $G$ , i.e.  $G_1 = G$  and  $G_{n+1} = [G_n, G]$ , the normal subgroup of  $G$  generated by elements of the form  $[g, h] = ghg^{-1}h^{-1}$  for all  $g \in G_n$  and  $h \in G$ . A group  $G$  is nilpotent of class  $n$  if  $G_{n+1} = 1$  but  $G_n \neq 1$ . We summarize some known facts about the group structures of  $\mathcal{H}(k)$  in the following lemma.

**LEMMA 2.3.** 1)  $\mathcal{H}(k)$  is torsion free and nilpotent of class  $k - 1$ .

2) Corresponding to a decomposition  $\mathcal{H}(k) = \mathcal{H}(k - 1) \times RF(k - 1)$ , we have

$$H(k)_n = \mathcal{H}(k - 1)_n \times RF(k - 1)_n.$$

3)  $\mathcal{H}(k)_{n-1}/\mathcal{H}(k)_n$  is a free abelian group of rank  $(n - 2)! \binom{k}{n}$ .

For  $\sigma \in \mathcal{H}(k)$ , its image in  $\mathcal{H}(k)/\mathcal{H}(k)_3$  can be described by  $\binom{k}{2} + \binom{k}{3}$  integers. These integers are linking numbers  $l_{ij}$ , for  $1 \leq i < j \leq k$ , and Milnor's triple linking numbers  $\mu(rst)$ , for  $1 \leq r < s < t \leq k$ . We want to have them defined precisely and understand how they change when  $\sigma$  is changed by a conjugation or a partial conjugation.

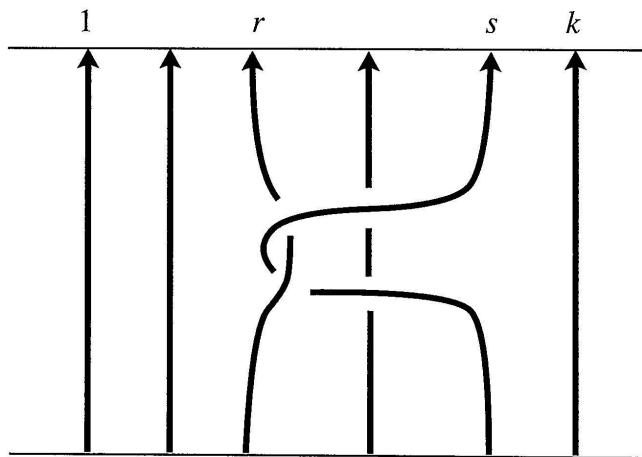


FIGURE 1  
The pure braid  $\tau_{rs}$

We will denote by  $\tau_{rs} = \tau_{sr}$ , for  $1 \leq r < s \leq k$ , the pure braid depicted in Figure 1. Let  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ . For  $1 \leq r < s < t \leq k$ , after deleting

all components other than the  $r, s, t$ -th components,  $\sigma$  can be written in the following normal form

$$(2) \quad \sigma = \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta,$$

where  $\alpha = l_{rs}$ ,  $\beta = l_{rt}$ ,  $\gamma = l_{st}$ . By definition, we have  $\delta = \mu(rst)$  for  $\sigma \in \mathcal{H}(k)$ .

LEMMA 2.4. *In  $\mathcal{H}(k)/\mathcal{H}(k)_3$ , if  $r', s', t'$  is a permutation of  $r, s, t$  and  $\epsilon$  is the sign of the permutation, then*

$$[\tau_{r't'}, \tau_{s't'}] = [\tau_{rt}, \tau_{st}]^\epsilon.$$

Furthermore, we have

$$[\tau_{rt}^\eta, \tau_{st}] = [\tau_{rt}, \tau_{st}]^\eta.$$

This lemma is useful in the following calculation and its proof is straightforward.

To understand how  $\mu(rst)$  changes under the conjugation, we only need to calculate the conjugation of  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  under the normal form (2) by  $\tau_{rs}, \tau_{rt}, \tau_{st}$ . This calculation is straightforward:

$$\begin{aligned} \tau_{rs} \sigma \tau_{rs}^{-1} &= \tau_{rs} \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta \tau_{rs}^{-1} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rs}, \tau_{rt}]^\beta [\tau_{rs}, \tau_{st}]^\gamma [\tau_{rt}, \tau_{st}]^\delta \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\beta-\gamma}; \\ \tau_{rt} \sigma \tau_{rt}^{-1} &= \tau_{rt} \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta \tau_{rt}^{-1} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{rs}]^\alpha [\tau_{rt}, \tau_{st}]^\gamma [\tau_{rt}, \tau_{st}]^\delta \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\alpha+\gamma}; \\ \tau_{st} \sigma \tau_{st}^{-1} &= \tau_{st} \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta \tau_{st}^{-1} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{st}, \tau_{rs}]^\alpha [\tau_{st}, \tau_{rt}]^\beta [\tau_{rt}, \tau_{st}]^\delta \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\alpha-\beta}. \end{aligned}$$

We summarize the calculation into the following lemma.

LEMMA 2.5. *The change of  $\mu(rst)$  under a conjugation is given by*

$$\text{Conjugation by } \tau_{rs}: \quad \mu(rst) \rightarrow \mu(rst) + l_{rt} - l_{st};$$

$$\text{Conjugation by } \tau_{rt}: \quad \mu(rst) \rightarrow \mu(rst) - l_{rs} + l_{st};$$

$$\text{Conjugation by } \tau_{st}: \quad \mu(rst) \rightarrow \mu(rst) + l_{rs} - l_{rt}.$$

Furthermore,  $\mu(rst)$  will not change under a conjugation by  $\tau_{ij}$  where  $\{i, j\}$  and  $\{r, s, t\}$  have at most one element in common.

The calculation of partial conjugations is slightly more complicated. We will start with partial conjugations by  $\tau_{rt}$  and  $\tau_{st}$ . These two operations are denoted by  $\mathbf{t}^r$  and  $\mathbf{t}^s$ , respectively. For  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  under the normal form (2), we have:

$$\begin{aligned}\sigma &\xrightarrow{\mathbf{t}^r} \tau_{rs}^\alpha \tau_{rt} \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta \tau_{rt}^{-1} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\gamma}; \\ \sigma &\xrightarrow{\mathbf{t}^s} \tau_{rs}^\alpha \tau_{st} \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta \tau_{st}^{-1} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\beta}.\end{aligned}$$

To calculate partial conjugations by  $\tau_{rs}$  and  $\tau_{ts}$ , which are denoted by  $\mathbf{s}^r$  and  $\mathbf{s}^t$ , respectively, we need to rewrite  $\sigma$  as follows:

$$\sigma = \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta = \tau_{rt}^\beta \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta}.$$

Then, we have:

$$\begin{aligned}\sigma &\xrightarrow{\mathbf{s}^r} \tau_{rt}^\beta \tau_{rs} \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta} \tau_{rs}^{-1} \\ &= \tau_{rt}^\beta \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta+\gamma} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\gamma}; \\ \sigma &\xrightarrow{\mathbf{s}^t} \tau_{rt}^\beta \tau_{ts} \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta} \tau_{ts}^{-1} \\ &= \tau_{rt}^\beta \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta-\alpha} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\gamma}.\end{aligned}$$

Similarly, to calculate partial conjugations  $\mathbf{r}^s$  and  $\mathbf{r}^t$ , we first rewrite  $\sigma$ :

$$\sigma = \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta = \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma}.$$

Then, we have

$$\begin{aligned}\sigma &\xrightarrow{\mathbf{r}^s} \tau_{st}^\gamma \tau_{sr} \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma} \tau_{sr}^{-1} \\ &= \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma+\beta} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\beta}; \\ \sigma &\xrightarrow{\mathbf{r}^t} \tau_{st}^\gamma \tau_{tr} \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma} \tau_{tr}^{-1} \\ &= \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma-\alpha} \\ &= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\alpha}.\end{aligned}$$

We summarize the previous calculation into the following lemma.

LEMMA 2.6. *The change of  $\mu(rst)$  under a partial conjugation is given by*

$$\begin{aligned} \mathbf{t}^r: & \mu(rst) \rightarrow \mu(rst) + l_{st}; \\ \mathbf{t}^s: & \mu(rst) \rightarrow \mu(rst) - l_{rt}; \\ \mathbf{s}^r: & \mu(rst) \rightarrow \mu(rst) - l_{st}; \\ \mathbf{s}^t: & \mu(rst) \rightarrow \mu(rst) + l_{rs}; \\ \mathbf{r}^s: & \mu(rst) \rightarrow \mu(rst) + l_{rt}; \\ \mathbf{r}^t: & \mu(rst) \rightarrow \mu(rst) - l_{rs}. \end{aligned}$$

Furthermore, a partial conjugation by  $\mathbf{i}^j$  will not change  $\mu(rst)$  if  $\{i, j\}$  and  $\{r, s, t\}$  have at most one element in common.

For a given string link  $\sigma \in \mathcal{H}(k)$ , we will think of the whole collection  $\{\mu(rst); 1 \leq r < s < t \leq k\}$  as an element in  $\mathbf{Z}^{\binom{k}{3}}$ . Then the conjugations and partial conjugations act on  $\mathbf{Z}^{\binom{k}{3}}$  by translations. We will abuse the notation by using the same symbol to denote both a translation operation and the corresponding translation vector. Thus, a translation operation  $T: V \rightarrow V$  on a vector space  $V$  is given by  $T(v) = v + T$ , for all  $v \in V$  and a fixed  $T \in V$ . If  $T_1$  and  $T_2$  are two translations, we have

$$(T_1 \cdot T_2)(v) = v + T_1 + T_2, \quad \text{for all } v \in V.$$

The following two theorems follow directly from Lemmas 2.5 and 2.6.

THEOREM 2.7. *The translation operation on  $\mathbf{Z}^{\binom{k}{3}}$  given by the conjugation of  $\tau_{ij}$  is the same as the composition of the translation operations given by the partial conjugations  $\mathbf{i}^j$  and  $\mathbf{j}^i$ , i.e. it is equal to  $\mathbf{i}^j + \mathbf{j}^i$ .*

THEOREM 2.8. *The translation operations  $\mathbf{i}^j$  satisfy the following relations:*

$$\sum_{j \neq i} \mathbf{j}^i = 0, \quad \sum_{j \neq i} l_{ij} \mathbf{i}^j = 0$$

for all  $i = 1, 2, \dots, k$ .

String links are oriented in the sense that each component is given an orientation from the bottom to the top. See Figure 1. Reversing the orientation on each component of a string link defines a bijection

$$\sigma \mapsto \bar{\sigma}: \mathcal{H}(k) \rightarrow \mathcal{H}(k).$$

This bijection is an anti-homomorphism:  $\overline{\sigma_1\sigma_2} = \overline{\sigma_2}\overline{\sigma_1}$ . This bijection induces an operation on  $\mathbf{Z}^{(k)}_{(3)}$ .

**THEOREM 2.9.** *The operation on  $\mathbf{Z}^{(k)}_{(3)}$  induced by reversing the orientation of each component of a string link is to change each  $\mu(rst)$  to  $-\mu(rst)$  followed by the translation operation*

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs}l_{rt} + l_{rs}l_{st} - l_{rt}l_{st}.$$

*Proof.* Consider the normal form (2) of  $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$  in the  $r, s, t$ -th components. The normal form for  $\overline{\sigma}$  is obtained as follows:

$$\begin{aligned} \overline{\sigma} &= [\tau_{rt}, \tau_{st}]^{-\delta} \tau_{st}^{\gamma} \tau_{rt}^{\beta} \tau_{rs}^{\alpha} \\ &= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{-\delta - \alpha\beta + \alpha\gamma - \beta\gamma}. \end{aligned}$$

Thus the operation on  $\mathbf{Z}^{(k)}_{(3)}$  induced by  $\sigma \mapsto \overline{\sigma}$  is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs}l_{rt} + l_{rs}l_{st} - l_{rt}l_{st}. \quad \square$$

### 3. CONSTRUCTION OF THE INVARIANT

By Theorems 2.2 and 2.7, we shall look for polynomials in  $l_{ij}$  and  $\mu(rst)$  invariant under the translation operations on  $\{\mu(rst)\} \in \mathbf{Z}^{(k)}_{(3)}$  induced by partial conjugations. There are  $k(k-1)$  partial conjugations altogether and their induced translations subject to  $2k$  linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of  $\{l_{ij}\}$ , the sublattice of  $\mathbf{Z}^{(k)}_{(3)}$  generated by the translation vectors of the partial conjugations will be of dimension no larger than  $k(k-1) - 2k = k^2 - 3k$ .

**LEMMA 3.1.** *For  $k > 3$ , the  $2k$  equations in Theorem 2.8 are linearly independent for generic values of  $\{l_{ij}\}$ .*

*Proof.* We write the two sets of equations in Theorem 2.8 as follows:

$$\begin{aligned} \mathbf{1}^i + \mathbf{2}^i + \cdots + \mathbf{j}^i + \cdots + \mathbf{k}^i &= 0, \quad j \neq i; \\ l_{i1}\mathbf{i}^1 + l_{i2}\mathbf{i}^2 + \cdots + l_{ij}\mathbf{i}^j + \cdots + l_{ik}\mathbf{i}^k &= 0, \quad j \neq i, \end{aligned}$$

for each  $i = 1, 2, \dots, k$ .

For generic values of  $\{l_{ij}\}$ , using the first  $k-1$  equations from the first set of  $k$  equations, we can solve for  $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^{k-1}$ . Similarly, we can solve