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4.2 Concluding remarks

It would be interesting to provide discrete analogues of other "4-vertex type" theorems known in the smooth case, and to find their specifically discrete proofs. We give two examples.

The following statement is a discrete version of the celebrated Möbius theorem (in dimension 2, "flattening" means "inflection") – see [9]:

An embedded non-contractible closed polygon in $\mathbf{RP}^2$ *has at least* 3 *flattenings.*

The notion of flattening for a polygonal line extends, in an obvious way, from $\mathbf{RP}^d$ to the sphere S^d. One has the following statement:

An embedded closed polygon in S^2 *bisecting the area has at least* 4 *flattenings.*

In the smooth case this was proved by B. Segre [14] and by V. Arnold (see [1, 2]).

We are confident that these statements hold true and can be proved in a similar way as in the smooth case. However, a detailed discussion would go beyond the limits of this article.

In conclusion, let us formulate a conjecture. For $k \geq d+2$ the following statement is stronger than Theorem 3.11.

Conjecture 4.2. *A strictly convex polygon in* $\mathbf{RP}^d$ *that intersects a hyperplane with multiplicity* k *has at least* k *flattenings.*

In the smooth case this is precisely Barner's result in full generality [3]. Conjecture 4.2 would imply strengthenings of Theorems 2.2, 2.6 and 2.10 – see [15] for the smooth case. For instance, the following result would hold.

Let X *and* Y *be two* n*-tuples of points in* $\mathbf{RP}^1$ *(see Section 2.3). If the closed broken line* $\big((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\big)$ *in* $\mathbf{RP}^1 \times \mathbf{RP}^1$ *intersects the graph of a projective transformation with multiplicity* k*, then there exist at least* k *extremal triples of indices.*

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