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4.2 CONCLUDING REMARKS

It would be interesting to provide discrete analogues of other “4-vertex type” theorems known in the smooth case, and to find their specifically discrete proofs. We give two examples.

The following statement is a discrete version of the celebrated Möbius theorem (in dimension 2, “flattening” means “inflection”) – see [9]:

An embedded non-contractible closed polygon in \mathbf{RP}^2 has at least 3 flattenings.

The notion of flattening for a polygonal line extends, in an obvious way, from \mathbf{RP}^d to the sphere S^d . One has the following statement:

An embedded closed polygon in S^2 bisecting the area has at least 4 flattenings.

In the smooth case this was proved by B. Segre [14] and by V. Arnold (see [1, 2]).

We are confident that these statements hold true and can be proved in a similar way as in the smooth case. However, a detailed discussion would go beyond the limits of this article.

In conclusion, let us formulate a conjecture. For $k \geq d + 2$ the following statement is stronger than Theorem 3.11.

CONJECTURE 4.2. *A strictly convex polygon in \mathbf{RP}^d that intersects a hyperplane with multiplicity k has at least k flattenings.*

In the smooth case this is precisely Barner’s result in full generality [3]. Conjecture 4.2 would imply strengthenings of Theorems 2.2, 2.6 and 2.10 – see [15] for the smooth case. For instance, the following result would hold.

Let X and Y be two n -tuples of points in \mathbf{RP}^1 (see Section 2.3). If the closed broken line $((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n))$ in $\mathbf{RP}^1 \times \mathbf{RP}^1$ intersects the graph of a projective transformation with multiplicity k , then there exist at least k extremal triples of indices.

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