Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

Autor: Ovsienko, V / Tabachnikov, S.

Kapitel: 4.2 CONCLUDING REMARKS

DOI: https://doi.org/10.5169/seals-65426

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

4.2 Concluding remarks

It would be interesting to provide discrete analogues of other "4-vertex type" theorems known in the smooth case, and to find their specifically discrete proofs. We give two examples.

The following statement is a discrete version of the celebrated Möbius theorem (in dimension 2, "flattening" means "inflection") – see [9]:

An embedded non-contractible closed polygon in $\mathbf{RP}^2$ *has at least* 3 *flattenings.*

The notion of flattening for a polygonal line extends, in an obvious way, from $\mathbf{RP}^d$ to the sphere S^d. One has the following statement:

An embedded closed polygon in S^2 *bisecting the area has at least* 4 *flattenings.*

In the smooth case this was proved by B. Segre [14] and by V. Arnold (see [1, 2]).

We are confident that these statements hold true and can be proved in a similar way as in the smooth case. However, a detailed discussion would go beyond the limits of this article.

In conclusion, let us formulate a conjecture. For $k \geq d+2$ the following statement is stronger than Theorem 3.11.

Conjecture 4.2. *A strictly convex polygon in* $\mathbf{RP}^d$ *that intersects a hyperplane with multiplicity* k *has at least* k *flattenings.*

In the smooth case this is precisely Barner's result in full generality [3]. Conjecture 4.2 would imply strengthenings of Theorems 2.2, 2.6 and 2.10 – see [15] for the smooth case. For instance, the following result would hold.

Let X *and* Y *be two* n*-tuples of points in* $\mathbf{RP}^1$ *(see Section 2.3). If the closed broken line* $\big((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\big)$ *in* $\mathbf{RP}^1 \times \mathbf{RP}^1$ *intersects the graph of a projective transformation with multiplicity* k*, then there exist at least* k *extremal triples of indices.*

ACKNOWLEDGMENTS. This work was supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach). We are grateful to the Mathematisches Forschungsinstitut at Oberwolfach for the creative atmosphere. The second author is also grateful to the Max-Planck Institut in Bonn for its hospitality. The second author was supported by an NSF grant.

REFERENCES

[1] ARNOLD, V. *Topological Invariants of Plane Curves and Caustics.* University Lecture Series 5, AMS 1994.

[2] —— Topological problems in the theory of wave propagation. *Russian Math. Surveys 51:1* (1996), 1–47.

[3] BARNER, M. Über die Mindestanzahl stationärer Schmiegebenen bei geschlossenen streng-konvexen Raumkurven. *Abh. Math. Sem. Univ. Hamburg 20* (1956), 196–215.

[4] BLASCHKE, W. *Vorlesungen über Differentialgeometrie*, Vol. 2. Springer-Verlag, 1923.

[5] CONNELY, R. Rigidity. In *Handbook of Convex Geometry*, 223–272. North-Holland, 1993.

[6] DUVAL, C. and V. OVSIENKO. Schwarzian derivative and Lorentzian world lines. *Funct. Anal. Appl. 34* (2000), 69–72.

[7] GHYS, E. *Cercles osculateurs et géométrie lorentzienne.* Talk at the Journée Inaugurale du CMI, 1995, Marseille.

[8] GUIEU, L., E. MOURRE and V. OVSIENKO. Theorem on six vertices of a plane curve via Sturm theory. *The Arnold-Gelfand Math. Seminars.* Birkhäuser, 1997, 257–266.

[9] MÖBIUS, A. F. Über die Grundformen der Linien der dritten Ordnung. *Abhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften, Math.-Phys. Klasse I* (1852), 1–82.

[10] MUKHOPADHYAYA, S. New methods in the geometry of a plane arc. *Bull. Calcutta Math. Soc. 1* (1909), 31–37.

[11] OVSIENKO, V. and S. TABACHNIKOV. Sturm theory, Ghys theorem on zeroes of the Schwarzian derivative and flattening of Legendrian curves. *Selecta Math. (N. S.) 2* (1996), 297–307.

[12] SEDYKH, V. A theorem on four support vertices of a polygonal line. *Funct. Anal. Appl. 30* (1996), 216–218.

[13] —— Discrete versions of the four-vertex theorem. *Amer. Math. Soc. Transl. Ser. 2, 180* (1997), 197–207.

[14] SEGRE, B. Alcune proprietà differenziali in grande delle curve chiuse sghembe. *Rend. Mat. (6) 1* (1968), 237–297.

[15] TABACHNIKOV, S. On zeroes of the Schwarzian derivative. *Amer. Math. Soc. Transl. Ser. 2, 180* (1997), 229–239.

[16] —— A four-vertex theorem for polygons. *Amer. Math. Monthly 107* (2000), 830–833.