Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

Autor: Ovsienko, V / Tabachnikov, S.

Kapitel: 4. Applications of the main theorem

DOI: https://doi.org/10.5169/seals-65426

where the dots indicate a linear combination of the remaining vectors. It follows from (3.6) that

$$\Delta_{n-d+i} = (-1)^{i-1}\, b\, \Delta'_{n-d+i}, \quad \Delta_{n-d+i+1} = (-1)^{d-i}\, a\, \Delta'_{n-d+i}\,. \tag{3.8}$$

It is time to use the strict convexity of P. Let H be a hyperplane in $\mathbf{RP}^d$ through $d-1$ vertices $V_{n-d+i+1}, \ldots, \widehat{V}_{n+1}, \ldots, V_{n+i}$ which intersects P with multiplicity $d-1$, and let $\widetilde{H}$ be its lifting to $\mathbf{R}^{d+1}$. Choose a linear function φ in $\mathbf{R}^{d+1}$ vanishing on $\widetilde{H}$ and such that $\varphi(\widetilde{V}_{n+1}) > 0$. We claim that

$$(-1)^{d-i}\varphi(\widetilde{V}_{n-d+i}) > 0 \quad \text{and} \quad (-1)^{i-1}\varphi(\widetilde{V}_n) > 0\,. \tag{3.9}$$

Indeed, by Lemma 3.3, the intersection multiplicities of $\widetilde{H}$ with the polygonal lines $(\widetilde{V}_{n-d+i}, \ldots, \widetilde{V}_{n+1})$ and $(\widetilde{V}_{n+1}, \ldots, \widetilde{V}_{n+i+1})$ are at least $d-i$ and $i-1$, respectively. Since H intersects P with multiplicity $d-1$, the above two multiplicities are indeed equal to $d-i$ and $i-1$. The inequalities (3.9) now readily follow from Lemma 3.5.

Finally, we evaluate φ on (3.7):

$$\varphi(\widetilde{V}_{n+1}) = a\,\varphi(\widetilde{V}_{n-d+i}) + b\,\varphi(\widetilde{V}_{n+i+1})\,.$$

It follows from (3.9) and the inequality $\varphi(\widetilde{V}_{n+1}) > 0$ that at least one of the numbers $(-1)^{i-1}b$ and $(-1)^{d-i}a$ is positive. In view of (3.8), Lemma 3.12 follows. □

Thus Theorem 3.11 is also proved. □

REMARK 3.13. Strict convexity is necessary for the existence of $d+1$ flattenings. One can easily construct a closed polygon without any flattenings and even C^0-approximate an arbitrary closed smooth curve by such polygons. In the smooth case such an approximation is well known: given a curve γ_0, the approximating one, γ, spirals around in a tubular neighbourhood of γ_0. In the polygonal case we take a sufficiently fine straightening of γ.

4. APPLICATIONS OF THE MAIN THEOREM

4.1 PROOF OF THEOREMS 2.2, 2.6 AND 2.10

Now we prove the results announced in Section 2. The idea is the same in all three cases and is precisely that of Barner's proof of the smooth versions of these theorems – see [3] and also [15]. We will consider Theorem 2.6 in detail, indicating the necessary changes in the other two cases.

Let P be as in Theorem 2.6. We consider the Veronese mapping $\mathcal{V}\colon \mathbf{RP}^2 \to \mathbf{RP}^5$ of degree 2:

$$\mathcal{V}\colon (x:y:z) \mapsto (x^2 : y^2 : z^2 : xy : yz : zx). \tag{4.1}$$

The image $\mathcal{V}(P)$ is a piecewise smooth curve. Every edge is homotopic to a straight segment, with the endpoints $\mathcal{V}(V_i), \mathcal{V}(V_{i+1})$ fixed, and we obtain a polygon Q in $\mathbf{RP}^5$. Assume first that Q is in general position.

LEMMA 4.1. *A quintuple $\mathcal{V}(V_i), \dots, \mathcal{V}(V_{i+4})$ is a flattening of Q if and only if $(V_i, \dots, V_{i+4})$ is an extremal quintuple of vertices of P.*

Proof. The Veronese map establishes a one-to-one correspondence between conics in $\mathbf{RP}^2$ and hyperplanes in $\mathbf{RP}^5$: the image of a conic is the intersection of a hyperplane with the quadric surface $\mathcal{V}(\mathbf{RP}^2)$. Since $\mathcal{V}$ is an embedding, the points V_{i-1} and V_{i+5} lie on one side of the conic through $(V_i, \dots, V_{i+4})$ if and only if the points $\mathcal{V}(V_{i-1})$ and $\mathcal{V}(V_{i+5})$ lie on one side of the corresponding hyperplane. $\square$

Next we show that the polygon Q is strictly convex. Given 4 indices i_1, i_2, i_3, i_4, we consider two lines in $\mathbf{RP}^2$: (V_{i_1}, V_{i_2}) and (V_{i_3}, V_{i_4}); the union of these lines is a conic that does not meet P any more. The corresponding hyperplane in $\mathbf{RP}^5$ contains the vertices $\mathcal{V}(V_{i_1}), \mathcal{V}(V_{i_2}), \mathcal{V}(V_{i_3}), \mathcal{V}(V_{i_4})$ and intersects Q with multiplicity 4.

Theorem 2.6 now follows from Theorem 3.11 for $d = 5$, provided Q is in general position. Otherwise, we replace P by a convex polygon P', close to P, such that the corresponding polygon Q' is in general position. Then, as above, P' has at least 6 extremal quintuples of vertices, and therefore so does P. This completes the proof.

To prove Theorems 2.2 and 2.10, one replaces the map (4.1) by the Veronese map $\mathcal{V} : \mathbf{RP}^2 \to \mathbf{RP}^3$

$$\mathcal{V} : (x:y:z) \mapsto (x^2 + y^2 : z^2 : yz : zx)$$

and by the Segre map $\mathcal{S}\colon \mathbf{RP}^1 \times \mathbf{RP}^1 \to \mathbf{RP}^3$

$$\mathcal{S}\colon ((x_1 : y_1), (x_2 : y_2)) \mapsto (x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2),$$

respectively. The proofs of strict convexity for the corresponding polygons Q reproduce those in the smooth case (see [15]).

4.2 CONCLUDING REMARKS

It would be interesting to provide discrete analogues of other "4-vertex type" theorems known in the smooth case, and to find their specifically discrete proofs. We give two examples.

The following statement is a discrete version of the celebrated Möbius theorem (in dimension 2, "flattening" means "inflection") – see [9]:

An embedded non-contractible closed polygon in $\mathbf{RP}^2$ *has at least* 3 *flattenings.*

The notion of flattening for a polygonal line extends, in an obvious way, from $\mathbf{RP}^d$ to the sphere S^d. One has the following statement:

An embedded closed polygon in S^2 *bisecting the area has at least* 4 *flattenings.*

In the smooth case this was proved by B. Segre [14] and by V. Arnold (see [1, 2]).

We are confident that these statements hold true and can be proved in a similar way as in the smooth case. However, a detailed discussion would go beyond the limits of this article.

In conclusion, let us formulate a conjecture. For $k \geq d+2$ the following statement is stronger than Theorem 3.11.

CONJECTURE 4.2. *A strictly convex polygon in* $\mathbf{RP}^d$ *that intersects a hyperplane with multiplicity* k *has at least* k *flattenings.*

In the smooth case this is precisely Barner's result in full generality [3]. Conjecture 4.2 would imply strengthenings of Theorems 2.2, 2.6 and 2.10 – see [15] for the smooth case. For instance, the following result would hold.

Let X *and* Y *be two* n*-tuples of points in* $\mathbf{RP}^1$ *(see Section 2.3). If the closed broken line* $\big((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\big)$ *in* $\mathbf{RP}^1 \times \mathbf{RP}^1$ *intersects the graph of a projective transformation with multiplicity* k*, then there exist at least* k *extremal triples of indices.*

ACKNOWLEDGMENTS. This work was supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach). We are grateful to the Mathematisches Forschungsinstitut at Oberwolfach for the creative atmosphere. The second author is also grateful to the Max-Planck Institut in Bonn for its hospitality. The second author was supported by an NSF grant.

REFERENCES

[1] ARNOLD, V. *Topological Invariants of Plane Curves and Caustics.* University Lecture Series 5, AMS 1994.

[2] —— Topological problems in the theory of wave propagation. *Russian Math. Surveys 51:1* (1996), 1–47.

[3] BARNER, M. Über die Mindestanzahl stationärer Schmiegebenen bei geschlossenen streng-konvexen Raumkurven. *Abh. Math. Sem. Univ. Hamburg 20* (1956), 196–215.

[4] BLASCHKE, W. *Vorlesungen über Differentialgeometrie*, Vol. 2. Springer-Verlag, 1923.

[5] CONNELY, R. Rigidity. In *Handbook of Convex Geometry*, 223–272. North-Holland, 1993.

[6] DUVAL, C. and V. OVSIENKO. Schwarzian derivative and Lorentzian world lines. *Funct. Anal. Appl. 34* (2000), 69–72.

[7] GHYS, E. *Cercles osculateurs et géométrie lorentzienne.* Talk at the Journée Inaugurale du CMI, 1995, Marseille.

[8] GUIEU, L., E. MOURRE and V. OVSIENKO. Theorem on six vertices of a plane curve via Sturm theory. *The Arnold-Gelfand Math. Seminars.* Birkhäuser, 1997, 257–266.

[9] MÖBIUS, A. F. Über die Grundformen der Linien der dritten Ordnung. *Abhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften, Math.-Phys. Klasse I* (1852), 1–82.

[10] MUKHOPADHYAYA, S. New methods in the geometry of a plane arc. *Bull. Calcutta Math. Soc. 1* (1909), 31–37.

[11] OVSIENKO, V. and S. TABACHNIKOV. Sturm theory, Ghys theorem on zeroes of the Schwarzian derivative and flattening of Legendrian curves. *Selecta Math. (N. S.) 2* (1996), 297–307.

[12] SEDYKH, V. A theorem on four support vertices of a polygonal line. *Funct. Anal. Appl. 30* (1996), 216–218.

[13] —— Discrete versions of the four-vertex theorem. *Amer. Math. Soc. Transl. Ser. 2, 180* (1997), 197–207.

[14] SEGRE, B. Alcune proprietà differenziali in grande delle curve chiuse sghembe. *Rend. Mat. (6) 1* (1968), 237–297.

[15] TABACHNIKOV, S. On zeroes of the Schwarzian derivative. *Amer. Math. Soc. Transl. Ser. 2, 180* (1997), 229–239.

[16] —— A four-vertex theorem for polygons. *Amer. Math. Monthly 107* (2000), 830–833.