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6.6 EXAMPLES

Keeping the notations of the previous section, we shall illustrate Theorem 17.

a. TOTALLY GEODESIC TRANSFORM. As in Section 4.1 **a**, let $X = G/K$ be a Riemannian symmetric space of the noncompact type and $y_o = \text{Exp } \mathfrak{s}$ the origin in the dual space $Y = G/H$. By (3) we have $\mathfrak{k} + \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{s}$, therefore Theorem 17 (i) applies with $\mathfrak{t} = \mathfrak{s}^\perp$, the orthogonal of \mathfrak{s} in \mathfrak{p} .

b. HOROCYCLE TRANSFORM. Again $X = G/K$ is a Riemannian symmetric space of the noncompact type (see Notations, **d**), but the dual space is now the space of horocycles $Y = G/MN$. We recall Harish-Chandra's isomorphism of algebras ([9], p. 306)

$$\Gamma : \mathbf{D}(X) \longrightarrow \mathbf{D}(A)^W,$$

where $\mathbf{D}(A)^W$ is the subalgebra of W -invariant differential operators in $\mathbf{D}(A)$. The definition of Γ will be recalled during the next proof.

PROPOSITION 18. *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$w(x, a) = a^\rho R_a^* v(x) = a^\rho \int_K v(gkaN) dk$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x, a) = \Gamma(P)_{(a)} w(x, a), \quad P \in \mathbf{D}(X), x \in X, a \in A.$$

Proof. Theorem 17 (ii) applies here with $T = A$, the abelian subgroup from the Iwasawa decomposition $G = KAN$; indeed $\mathfrak{k} + \mathfrak{h} = \mathfrak{k} + \mathfrak{m} + \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{n}$, and $\mathfrak{g} = (\mathfrak{k} \oplus \mathfrak{n}) \oplus \mathfrak{a}$, $[\mathfrak{a}, \mathfrak{h}] \subset [\mathfrak{a}, \mathfrak{m}] + [\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n} \subset \mathfrak{h}$. By (31) we thus have

$$(32) \quad P_{(x)} R_a^* v(x) = D'_{(a)} R_a^* v(x),$$

where $D \in \mathbf{D}(G)^K$ is related to P by (28) and $D' \in \mathbf{D}(A)$ was characterized by

$$(33) \quad D - D' \in \mathfrak{k}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{n}.$$

To compare D' and $\Gamma(P)$ we recall that $\Gamma(P) = a^{-\rho} D_a \circ a^\rho$, where $D_a \in \mathbf{D}(A)$ is characterized by

$$(34) \quad D - D_a \in \mathfrak{n}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{k}.$$

Moreover $(Df)(a) = D_a(f(a))$ for $a \in A$, if $f \in C^\infty(G)$ is such that $f(ngk) = f(g)$ for any $g \in G, k \in K, n \in N$ ([9], p. 302 sq.).

Taking $u \in \mathcal{D}(G)$ we have, by a classical integral formula,

$$(35) \quad \int_G Df(g) \cdot u(g) dg = \int_{N \times A \times K} Df(a) \cdot u(nak) a^{-2\rho} dn da dk \\ = \int_{N \times A \times K} D_a f(a) \cdot u(nak) a^{-2\rho} dn da dk.$$

On the other hand, this integral can be written with the transpose operator tD as

$$\int_G Df(g) \cdot u(g) dg = \int_G f(g) {}^tDu(g) dg \\ = \int_A f(a) a^{-2\rho} da \int_{N \times K} ({}^tDu)(nak) dn dk.$$

But ${}^tD \in \mathbf{D}(G)^K$ therefore, for any $g \in G$,

$$\int_{N \times K} ({}^tDu)(ngk) dn dk = ({}^tD)_{(g)} \left(\int_{N \times K} u(ngk) dn dk \right).$$

The latter integral, as a function of g , is left N -invariant and right K -invariant so that

$$\int_{N \times K} ({}^tDu)(nak) dn dk = ({}^tD)_a \left(\int_{N \times K} u(nak) dn dk \right).$$

Since $({}^tD)_a = {}^t(D')$ obviously by (33) and (34), we obtain

$$\int_G Df(g) \cdot u(g) dg = \int_A D'(f(a)a^{-2\rho}) da \int_{N \times K} u(nak) dn dk \\ = \int_{N \times A \times K} (a^{2\rho} D' \circ a^{-2\rho}) f(a) \cdot u(nak) a^{-2\rho} dn da dk,$$

for any $f \in C^\infty(A)$ and any $u \in \mathcal{D}(G)$. Comparing with (35) it follows that

$$D_a = a^{2\rho} D' \circ a^{-2\rho}, \quad D' = a^{-\rho} \Gamma(P) \circ a^\rho,$$

whence the result by (32). \square

A slightly different proof can be obtained by decomposing the wave $a^\rho R_a^* v(gK)$ into *elementary horocycle waves* as follows. For $g \in G$ we denote by $A(g) \in A$ the A -component of g in the Iwasawa decompositions $G = NAK = ANK$ (we recall that A normalizes N), and by $K(g) \in K$ its K -component in the decompositions $G = KAN = KNA$.

PROPOSITION 19. (i) Given $f \in C^\infty(A)$ and $k \in K$, the function

$$w(gK, a) = a^{-\rho} f(A(k^{-1}g)a)$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x, a) = \Gamma(P)_{(a)} w(x, a), \quad P \in \mathbf{D}(X), \quad x \in X, \quad a \in A.$$

(ii) Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by

$$a^\rho R_a^* v(gK) = \int_K a^\rho v(gkaN) dk$$

is a solution of the same equations.

REMARKS. Part (i) is Proposition 8.5 in [12], p.118. Note that, k being fixed, the “wave surfaces” $A(k^{-1}g) = \text{constant}$ are parallel horocycles with the same normal $kM \in K/M$ (cf. [11], p.81). Indeed the equality $A(k^{-1}g) = a_o \in A$ is equivalent to $k^{-1}g \in a_oNK$, i.e. $g \cdot x_o \in ka_o \cdot y_o$.

If λ is a linear form on \mathfrak{a} and $f(a) = a^{i\lambda + \rho}$, the result (i) implies that $A(k^{-1}g)^{i\lambda + \rho}$ is, as a function of gK , an eigenfunction of all invariant operators $P \in \mathbf{D}(X)$; this is a fundamental result for harmonic analysis on X .

Part (ii) provides a simpler proof and a generalization of Proposition 8.6 in [12], p.118, where v was the Radon transform Ru of some $u \in \mathcal{D}(X)$. We refer to [12] or [13] for a detailed study of those multitemporal wave equations.

Proof of Proposition 19. (i) Both sides of the wave equation are invariant under the action of K on X ; we can therefore assume $k = e$. Now $w(gK, a) = a^{-\rho} f(A(g)a)$ is left N -invariant and right K -invariant as a function of g , and it will suffice to prove the result for $g = a \in A$.

By the decomposition (34) of D we have, for any $b \in A$,

$$D_{(g)} (f(A(g)b))|_{g=a} = (D_a)_{(a)} (f(ab)) = a^\rho \Gamma(P)_{(a)} (a^{-\rho} f(ab)) .$$

But $\Gamma(P)$ is an invariant differential operator on A , isomorphic to the additive group of a vector space, and we obtain

$$\begin{aligned} D_{(g)} (b^{-\rho} f(A(g)b))|_{g=a} &= a^\rho \Gamma(P)_{(a)} ((ab)^{-\rho} f(ab)) \\ &= a^\rho \Gamma(P)_{(b)} ((ab)^{-\rho} f(ab)) \\ &= \Gamma(P)_{(b)} (b^{-\rho} f(ab)) = \Gamma(P)_{(b)} (b^{-\rho} f(A(g)b))|_{g=a} . \end{aligned}$$

Thus (i) is proved for $g = a$.

(ii) Let $g \in G$, $k \in K$ and $k' = K(gk)$. Then $gk = k'a'n'$ with $a' \in A$ and $n' \in N$. It follows that $k'^{-1}g = a'n'k^{-1}$, therefore $a' = A(k'^{-1}g)$ and

$$gkaN = k'A(k'^{-1}g)aN.$$

For fixed g the map $k \mapsto K(gk) = k'$ is a diffeomorphism of K onto itself and, by the integral formula ([9], p. 197)

$$\int_K F(k') dk = \int_K A(k'^{-1}g)^{2\rho} F(k') dk',$$

we have

$$\begin{aligned} a^\rho R_a^* v(gK) &= a^\rho \int_K v(gkaN) dk \\ &= a^\rho \int_K v(k'A(k'^{-1}g)aN) dk \\ &= a^{-\rho} \int_K (A(k'^{-1}g)a)^{2\rho} v(k'A(k'^{-1}g)aN) dk'. \end{aligned}$$

By (i) applied to the functions $f(a) = a^{2\rho}v(k'aN)$, $k' \in K$, this is a solution of the wave equations. \square

COROLLARY 20 (Helgason). *If \mathfrak{g} has only one conjugacy class of Cartan subalgebras, there exists a differential operator $P \in \mathbf{D}(X)$ such that the horocycle Radon transform of $X = G/K$ is inverted by*

$$u(x) = PR^*Ru(x)$$

for $u \in \mathcal{D}(X)$, $x \in X$.

We prove it here by means of shifted transforms and wave equations; see [11], p. 116 for Helgason's original proof.

Proof. The assumption on \mathfrak{g} implies that, in the notation of (15), $C \cdot |c(\lambda)|^{-2}$ is a W -invariant polynomial on \mathfrak{a}^* . Let $P \in \mathbf{D}(X)$ be the corresponding operator under the isomorphism $\Gamma: \mathbf{D}(X) \rightarrow \mathbf{D}(A)^W$, so that $\Gamma(P)(i\lambda) = C \cdot |c(\lambda)|^{-2}$. By Theorem 13 and Proposition 19 (ii) (with $v = Ru$)

we have

$$\begin{aligned} u(x) &= \langle T_{(a)}, a^\rho R_a^* Ru(x) \rangle = \Gamma(D)_{(a)} (a^\rho R_a^* Ru(x)) \Big|_{a=e} \\ &= P_{(x)} (a^\rho R_a^* Ru(x)) \Big|_{a=e} = P_{(x)} R^* Ru(x). \quad \square \end{aligned}$$

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