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In particular, at the origin,

$$\begin{aligned}(2\pi)^k u(x_o) &= \left(\tau^{-1}\partial_\tau\right)^k \left(\tau^{-1-q'} Ru(t(\tau))\right)_{\tau=1} \\ &= \left(\partial_\tau^k + \cdots + (-1)^k (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t(\tau))\Big|_{\tau=1}.\end{aligned}$$

To switch over to derivatives with respect to t we note that, if $g(\tau) = f(t)$ with $\tau = (\cosh t)^{-1} = 1 - \frac{t^2}{2} + \cdots$, identification of Taylor expansions at $\tau = 1$, resp. $t = 0$, leads to

$$\left(-\frac{1}{2}\right)^k \frac{g^{(k)}(1)}{k!} = \frac{f^{(2k)}(0)}{(2k)!} + \cdots + a_k f''(0),$$

where dots are a sum of even derivatives of f multiplied by some rational coefficients (like a_k). Therefore

$$(-2\pi)^k u(x_o) = \left(\frac{2^k k!}{(2k)!}\partial_t^{2k} + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t)|_{t=0},$$

for any K-invariant $u \in \mathcal{D}(X)$, whence the claim by Section 6.2. □

6.4 The amusing formula generalized

a. To motivate the forthcoming generalizations of the amusing formula (12) and their applications to Radon inversion, we briefly recall the classical example of points and hyperplanes in the Euclidean space $X = \mathbf{R}^n$. Let (ω, p) be parameters for the hyperplane defined by the equation $\omega \cdot x = p$, where ω is a unit vector, p is a real number and $\cdot$ is the scalar product. Given $t \in \mathbf{R}$ and a point $x \in \mathbf{R}^n$, the parameters $(\omega, p) = (\omega, t + \omega \cdot x)$ define a hyperplane at distance $|t|$ from x, and

$$R_t^* v(x) = \int_{S^{n-1}} v(\omega, t + \omega \cdot x)\, d\omega$$

is the corresponding shifted dual Radon transform, where $v(\omega, p) = v(-\omega, -p)$ is an arbitrary smooth even function on $S^{n-1} \times \mathbf{R}$. Changing ω into $-\omega$ in the integral shows that $R_t^* v(x)$ is an even function of t.

Since $\sum \omega_i^2 = 1$ it is easily checked that

$$\left(\partial_t^2 - \Delta_x\right) v(\omega, t + \omega \cdot x) = 0,$$

where Δ_x is the Euclidean Laplace operator acting on x. Thus $R_t^* v(x)$, as a function of (x, t) in $\mathbf{R}^n \times \mathbf{R}$, is a solution of the wave equation, being an integral of the *elementary plane waves* $v(\omega, t+\omega\cdot x)$. More generally, for any positive integer k,

(19) $$\left(\partial_t^{2k} - \Delta_x^k\right) R_t^* v(x) = 0\,.$$

For odd n we have, by Theorem 8 with $n = 2k+1$, $d = 2k$ and $\varepsilon = 0$, the following inversion formula for the Radon transform on hyperplanes

(20) $$Cu(x) = \Delta_x^k R^* Ru(x)\,.$$

Putting $v = Ru$ in (19) and observing that $R^* = R_0^*$, we thus obtain a new inversion formula by means of the shifted dual transform

(21) $$Cu(x) = \partial_t^{n-1} R_t^* Ru(x)|_{t=0}\,.$$

Formula (21) might also be proved directly by the method of Section 6.2.

b. To extend formula (12) we first deal with the Laplace operator; general invariant operators will be considered in the next section.

Let G be a Lie group, K a compact subgroup and let L be the Laplace operator of the Riemannian manifold $X = G/K$ (cf. Notations, **b**). The operator L can be expressed by means of any orthonormal basis $X_1, \dots, X_n$ of $\mathfrak{p}$ as

$$Lf(gK) = \sum_{j=1}^{n} \partial_s^2 f\left(g \exp(sX_j)K\right)\big|_{s=0}\,,$$

with $f \in C^2(G/K)$, $g \in G$; indeed both sides are G-invariant operators on X which coincide at $g = e$.

Now let $Y = G/H$ where H is a Lie subgroup of G and, as before,

$$R^* v(gK) = \int_K v(gkH)\, dk\,, \quad R_t^* v(gK) = \int_K v(gktH)\, dk$$

for $v \in C^2(Y)$ and $g, t \in G$. Then

(22) $$LR^* v(gK) = \int_K \left(\sum_j \partial_s^2 v\left(g \exp(sX_j)kH\right)\big|_{s=0}\right) dk\,.$$

But $\sum X_j^2$ is a K-invariant element in the symmetric algebra of $\mathfrak{p}$ and it follows that, for any $\varphi \in C^2(\mathfrak{p})$, $k \in K$,

$$\sum_j \partial_s^2 \varphi\left(sX_j\right)\big|_{s=0} = \sum_j \partial_s^2 \varphi\left(s\left(k\cdot X_j\right)\right)\Big|_{s=0}\,.$$

Therefore k can be moved to the left of $\exp sX_j$ in (22) and we obtain

(23) $$LR^*v(x) = \sum_j \partial_s^2 R^*_{\exp sX_j} v(x)\Big|_{s=0}$$

for $v \in C^2(Y)$, $x \in X$. If $\mathfrak{h} \cap \mathfrak{p}$ is a nontrivial subspace of $\mathfrak{p}$ and the basis (X_j) contains a basis of this subspace, the sum in (23) only runs over an orthonormal basis of the orthogonal subspace $(\mathfrak{h} \cap \mathfrak{p})^\perp$, due to the right H-invariance of v.

We now give a more specific result for the *geodesic Radon transform*, in the notation of Section 4.1. If $\mathfrak{s}$ is a d-dimensional Lie triple system contained in $\mathfrak{p}$ and $y_o = \operatorname{Exp} \mathfrak{s}$ the corresponding totally geodesic submanifold of X, we take as Y the set of all $g \cdot y_o$ for $g \in G$. Then $Y = G/H$, where H is the subgroup of all $h \in G$ globally preserving y_o.

PROPOSITION 16. *Let X be one of the classical hyperbolic spaces $H^n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Assume $\mathfrak{s}$ is a $\mathbf{F}$-vector subspace of $\mathfrak{p}$ and let $T \in \mathfrak{p}$ be any unit vector orthogonal to $\mathfrak{s}$. For $v \in C^2(Y)$, the shifted dual geodesic transform $R^*_{\exp tT} v$ is then an even function of $t \in \mathbf{R}$ and, for $x \in X$,*

$$LR^*v(x) = (n-d)\, \partial_t^2 R^*_{\exp tT} v(x)\Big|_{t=0}$$

where n and d denote the real *dimension of X and $\mathfrak{s}$ respectively.*

In other words, the function $(x,t) \mapsto R^*_{\exp tT} v(x)$ is a solution at time $t = 0$ of the wave operator $L - (n-d)\partial_t^2$ on $X \times \mathbf{R}$.

Applying the proposition to $H^3(\mathbf{R})$ with $d = 2$ we obtain formula (12). Indeed, if $\varphi(t)$ is an even function of t, let ψ be defined by $\psi(\tau) = \varphi(t)$ with $\cosh t = 1/\tau$; then $-\psi'(1) = \varphi''(0)$.

EXAMPLE. By Theorem 8 the 2-geodesic transform on $X = H^n(\mathbf{R})$ can be inverted by means of a second order differential operator:

$$-2\pi(n-2)u = (L + n - 2)R^*Ru\,,$$

and Proposition 16 now yields the inversion formula

(24) $$-2\pi u = \left(\partial_t^2 + 1\right) R^*_{\exp tT} Ru\Big|_{t=0}\,,$$

where $u \in \mathcal{D}(X)$ and $T \in \mathfrak{p}$ is any unit vector orthogonal to $\mathfrak{s}$. Formula (24) also follows from Theorem 14 (ii) with $k = 1$, $q' = 0$.

Proof of Proposition 16. The point is to show that the group $K \cap H$ acts transitively on the unit sphere of $\mathfrak{s}^\perp$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$. For the scalar product $(T,V) = \mathrm{Re} \sum \bar{T}_i V_i$ on $\mathfrak{p}$ we have $(T, V\lambda) = (T\bar{\lambda}, V)$, $\lambda \in \mathbf{F}$, therefore $\mathfrak{s}^\perp$ is a $\mathbf{F}$-subspace of $\mathfrak{p}$.

An element k of $K \cap H$ is characterized by $k \in K$ and $k \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$, i.e. $k \cdot \mathfrak{s} = \mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of $\mathfrak{p}$ and $\mathfrak{s}$ as $\mathbf{F}$-vector spaces. Taking a $\mathbf{F}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, it follows that

$$K = U(1;\mathbf{F}) \times U(n';\mathbf{F}), \quad K \cap H = U(1;\mathbf{F}) \times U(d';\mathbf{F}) \times U(n'-d';\mathbf{F}).$$

But $U(n'-d';\mathbf{F})$ acts transitively on the unit sphere of $\mathbf{F}^{n'-d'}$, which implies our claim.

If $T, T' \in \mathfrak{s}^\perp$ are two unit vectors, there exists $k_o \in K \cap H$ such that $k_o \cdot T = T'$. Thus

$$\begin{aligned} R^*_{\exp tT'} v(gK) &= \int_K v\left(gkk_o \exp(tT) k_o^{-1} H\right) dk \\ &= \int_K v\left(gk \exp(tT) H\right) dk = R^*_{\exp tT} v(gK). \end{aligned}$$

In particular $R^*_{\exp tT} v$ is an even function of t.

Going back to (23), we now take as (X_j) an orthonormal $\mathbf{R}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. The $n-d$ basis vectors in $\mathfrak{s}^\perp$ give the same contribution to the right hand side, whereas the d vectors in $\mathfrak{s}$ generate one parameters subgroups of H and give no contribution; indeed $\exp tV \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$ for $V \in \mathfrak{s}$, since $\mathfrak{s}$ is a Lie triple system by Section 4.3 **c**. This completes the proof. □

6.5 Multitemporal waves

We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X = G/K$, $Y = G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the respective Lie algebras, and $\mathfrak{t}$ a vector subspace of $\mathfrak{g}$ such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let $K_1, \ldots, K_p$ be a basis of $\mathfrak{k}$, complemented by $H_1, \ldots, H_q \in \mathfrak{h}$ so that the K_i's and H_j's are a basis of $\mathfrak{k} + \mathfrak{h}$, and let $T_1, \ldots, T_r$ be a basis of $\mathfrak{t}$. We shall use the same notations for the corresponding left-invariant vector fields on G, e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with $f \in C^\infty(G)$, $g \in G$, $s \in \mathbf{R}$. We denote by $\mathbf{D}(G)$ the algebra of all left invariant differential operators on G, by $\mathbf{D}(G)^K$ the subalgebra of right