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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

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The general inversion formula (14) *for* R *thus follows from the special case* (13) *of* K*-invariant functions at the origin*, thanks to the shifted dual transform.

If X is an isotropic space, the above trick (replace u by u_g) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

6.3 EXAMPLES

a. HOROCYCLE TRANSFORM. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, **d**), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in X, resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o)\, dn$$

(integrating over M is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o)\, dk\,.$$

For K-invariant u the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o)\, dn = a^{-\rho} \mathcal{A}u(a)\,;$$

the *Abel transform* $\mathcal{A}$ is defined by this equality.

For K-invariant $u \in \mathcal{D}(X)$ we have $\mathcal{A}u \in \mathcal{D}(A)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. It is known from spherical harmonic analysis on X that the classical Fourier transform

$$\widehat{\mathcal{A}u}(\lambda) = \int_A a^{-i\lambda} \mathcal{A}u(a)\, da\,, \quad \lambda \in \mathfrak{a}^*\,,$$

coincides with the spherical transform of u, with the inversion formula ([9] p. 454)

$$u(x_o) = C \int_{\mathfrak{a}^*} \widehat{\mathcal{A}u}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda\,, \tag{15}$$

where C is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since

$C \cdot |c(\lambda)|^{-2}$ has polynomial growth on $\mathfrak{a}^*$ its Fourier transform is a tempered distribution T on $A = \exp \mathfrak{a}$ such that

$$u(x_o) = \langle T, \mathcal{A}u \rangle = \langle T_{(a)}, a^\rho Ru(a \cdot y_o) \rangle .$$

Thus T inverts the Abel transform at the origin. By (14) we obtain the next theorem.

THEOREM 13. *Let X be a Riemannian symmetric space of the noncompact type. Its horocycle Radon transform R can be inverted by*

$$u(x) = \langle T_{(a)}, a^\rho R_a^* Ru(x) \rangle, \ x \in X ,$$

for $u \in \mathcal{D}(X)$. The distribution $T_{(a)}$ (acting on the variable $a \in A$) is, up to a constant factor, the Fourier transform of $|c(\lambda)|^{-2}$.

REMARKS.

(i) This extends a result by Berenstein and Tarabusi [2] for $X = H^n(\mathbf{R})$, obtained by direct calculations.

(ii) Helgason's original inversion formula ([11], p. 116)

$$u(x) = R^* \Lambda \overline{\Lambda} Ru(x)$$

follows easily from Theorem 13. Indeed Helgason's operator $\Lambda\overline{\Lambda}$ is defined as follows ([11], p. 111). Given $v \in \mathcal{D}(Y)$ and $g = kan \in G$, multiply $v(g \cdot y_o) = v(ka \cdot y_o)$ by a^ρ, take the Fourier transform with respect to $a \in A$, multiply it by $C \cdot |c(\lambda)|^{-2}$ (an even function of λ), take the inverse Fourier transform, and multiply by $a^{-\rho}$; the result is $\Lambda\overline{\Lambda} v(g \cdot y_o)$. In other words

$$\Lambda\overline{\Lambda} v(g \cdot y_o) = \Lambda\overline{\Lambda} v(ka \cdot y_o) = a^{-\rho} \left(T * (a^\rho v)\right)(ka \cdot y_o) ,$$

where $*$ is the convolution on A with respect to a. Let b denote a variable in A; since T is even we have

$$\begin{aligned} \Lambda\overline{\Lambda} v(g \cdot y_o) &= a^{-\rho} \langle T_{(b)}, (ab)^\rho v(kab \cdot y_o) \rangle \\ &= \langle T_{(b)}, b^\rho v(kab \cdot y_o) \rangle = \langle T_{(b)}, b^\rho v(gb \cdot y_o) \rangle . \end{aligned}$$

Replacing v by Ru, g by gk and integrating with respect to $k \in K$ we obtain

$$\begin{aligned} R^* \Lambda\overline{\Lambda} Ru(g \cdot x_o) &= \int_K \langle T_{(b)}, b^\rho Ru(gkb \cdot y_o) \rangle dk \\ &= \left\langle T_{(b)}, b^\rho \int_K Ru(gkb \cdot y_o)\, dk \right\rangle = \langle T_{(b)}, b^\rho R_b^* Ru(g \cdot x_o) \rangle . \end{aligned}$$

By Theorem 13 this is $u(g \cdot x_o)$, as claimed.

(iii) Note that T is supported at the origin if and only if $|c(\lambda)|^{-2}$ is a polynomial, i.e. if the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below).

b. TOTALLY GEODESIC TRANSFORM ON CLASSICAL HYPERBOLIC SPACES. We retain the notation of Section 4.3 **c**.

THEOREM 14. *Let $X = H^m(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, be one of the classical hyperbolic spaces, let $\mathfrak{s}$ be any $\mathbf{F}$-vector subspace of $\mathfrak{p} = \mathbf{F}^m$, and T any unit vector orthogonal to $\mathfrak{s}$ in $\mathfrak{p}$.*

For the Radon transform defined by the totally geodesic submanifolds $y = g \cdot \operatorname{Exp} \mathfrak{s}$, of (real) dimension d, we have the following inversion formulas by means of shifted dual transforms, for $u \in \mathcal{D}(X)$ and $x \in X$.

(i) *If $d = 2k + 1$ is odd, $k \geq 0$,*

$$2^k \pi^{k+1} u(x) = \left(\sigma^{-1} \partial_\sigma\right)^{k+1} \int_0^\sigma \left(R^*_{\exp t(\tau)T} Ru(x)\right) (\sigma^2 - \tau^2)^{-1/2} d\tau \Big|_{\sigma=1},$$

where $t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$.

(ii) *If $d = 2k$ is even, $k \geq 1$, there exists a polynomial of degree k*

$$Q_k(\lambda) = \frac{2^k k!}{(2k)!} \lambda^k + \cdots + (q' + 1)(q' + 3) \cdots (q' + 2k - 1),$$

with rational coefficients (depending on k and $q' = \dim \mathfrak{s}_{2\alpha}$), such that

$$(-2\pi)^k u(x) = Q_k(\partial_t^2) \left(R^*_{\exp tT} Ru(x)\right)_{t=0}.$$

REMARKS. This extends a result proved by Helgason ([10], p. 144, or [14], p. 97) for $\mathbf{F} = \mathbf{R}$. In case (i), a look at the proof below shows that an arbitrary positive integer ℓ may be added to the exponents of $\sigma^{-1}\partial_\sigma$ and $\sigma^2 - \tau^2$; Helgason's result is obtained for $\ell = k$. From the proof of case (ii) we obtain for $k = 1, 2$

$$Q_1(\partial_t^2) = \partial_t^2 + q' + 1$$

$$Q_2(\partial_t^2) = \frac{1}{3}\partial_t^4 + \left(2q' + \frac{14}{3}\right) \partial_t^2 + (q' + 1)(q' + 3).$$

Our d is of course even whenever $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$. A comparison with Section 4.3 **c** shows that (except for $\mathbf{F} = \mathbf{R}$) the present assumption on $\mathfrak{s}$ is stronger than in Theorem 8.

Proof of Theorem 14. In order to use spherical coordinates on totally geodesic submanifolds of X, we need a lemma. As in Section 4.3 **c**, the matrices in $\mathfrak{p}$ can be identified to vectors $V = (V_1, \dots, V_m) \in \mathbf{F}^m$, and the scalar product of $T, V \in \mathfrak{p}$ is

$$(T, V) = \operatorname{Re}\left(\bar{T} \cdot V\right), \quad \text{with } \bar{T} \cdot V = \sum_{i=1}^{m} \bar{T}_i V_i .$$

Let $\| \ \|$ be the corresponding norm.

LEMMA 15. *Let $X = H^m(\mathbf{F})$ be a classical hyperbolic space.*

(i) *Let $T, V \in \mathfrak{p}$. In the geodesic triangle with vertices x_o (the origin of X), $\operatorname{Exp} T$ and $\exp T \cdot \operatorname{Exp} V$, the Riemannian lengths of the sides are $t = \|T\|$, $r = \|V\|$ and w given by*

$$\cosh^2 w = \left(\cosh t \cosh r + \frac{\sinh t}{t}\frac{\sinh r}{r}(T, V)\right)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\bar{T} \cdot V - (T, V)\right|\right)^2 .$$

(ii) *Let $\mathfrak{s} \subset \mathfrak{p}$ be a Lie triple system. If $T \in \mathfrak{p}$ is orthogonal to $\mathfrak{s}$, the totally geodesic submanifold $\exp T \cdot \operatorname{Exp} \mathfrak{s}$ is at distance $t = \|T\|$ from the origin.*

Proof. (i) The Riemannian distance from x_o to $\operatorname{Exp} T$ is $\|T\| = t$. Transforming x_o and $\operatorname{Exp} V$ by the isometry $\exp T \in G$ shows that the second side of the triangle has length r. The third side is $w = \|W\|$, where W is the unique $W \in \mathfrak{p}$ such that $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$, in other words

$$\exp W = (\exp T \exp V)\, k$$

for some $k \in K$. The map $g \mapsto g\theta(g)^{-1}$, where θ is the Cartan involution of G, transforms this equality into

$$\exp 2W = \exp T \exp 2V \exp T .$$

By elementary matrix computations $T^3 = t^2 T$, and the exponential is

$$\exp T = I + \frac{\sinh t}{t} T + \frac{\cosh t - 1}{t^2} T^2 ,$$

where I is the unit matrix. Now $\operatorname{tr} T = 0$ and $\operatorname{tr} T^2 = 2t^2$ is real, so that taking the traces we obtain

$$\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2T \exp 2V) ;$$

indeed $\operatorname{Re}\operatorname{tr}(gg') = \operatorname{Re}\operatorname{tr}(g'g)$ for $g, g' \in G$, even when $\mathbf{F} = \mathbf{H}$.

Taking account of

$$\operatorname{Re}\operatorname{tr} TV = 2(T,V), \qquad \operatorname{tr} T^2V = \operatorname{tr} TV^2 = 0,$$
$$\operatorname{Re}\operatorname{tr} T^2V^2 = t^2r^2 + \left|\bar{T}\cdot V\right|^2,$$

the expression of $\cosh w$ follows after some elementary calculations.

(ii) Let $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$. By (i) with $V \in \mathfrak{s}$ and $(T,V) = 0$, the distance w of the origin to the point $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$ of y is given by

$$\cosh^2 w = (\cosh t \cosh r)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\bar{T}\cdot V\right|\right)^2 .$$

Therefore $w \geq t$, with equality if and only if $V = 0$, and $\operatorname{Exp} T$ is the unique point of y closest to x_o (geodesic orthogonal projection of the origin on y). The lemma is proved. $\square$

Going back to Theorem 14, let $g \in G$ and let $y = g \cdot \operatorname{Exp} \mathfrak{s}$ be an arbitrary given totally geodesic submanifold, element of Y. The minimum distance between y and the origin x_o is obtained at a point $\operatorname{Exp} T \in y$, with $T \in \mathfrak{p}$. In particular there exists $V \in \mathfrak{s}$ such that $\operatorname{Exp} T = g \cdot \operatorname{Exp} V$, i.e. $(\exp T)k = g \exp V$ for some $k \in K$. But $\operatorname{Exp} \mathfrak{s}$ is globally invariant under the action of $\exp V$, so that $y = (\exp T)k \cdot \operatorname{Exp} \mathfrak{s} = \exp T \cdot \operatorname{Exp}(k \cdot \mathfrak{s})$. Changing notation, we may write $\mathfrak{s}$ for $k \cdot \mathfrak{s}$ and $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$.

Let $V \in \mathfrak{s}$. On the geodesic $\exp T \cdot \operatorname{Exp} sV$, $s \in \mathbf{R}$, contained in y, the minimum distance to x_o is obtained for $s = 0$. By Lemma 15 (i) with sV instead of V, this implies $(T,V) = 0$ so that T is orthogonal to $\mathfrak{s}$ and Lemma 15 (ii) applies.

Besides, if we assume $\mathfrak{s}$ is a $\mathbf{F}$-vector subspace of $\mathfrak{p}$ therefore a Lie triple system (Section 4.3 **c**), the vector T must be orthogonal to all $V\lambda$, $V \in \mathfrak{s}$, $\lambda \in \mathbf{F}$, whence $\bar{T}\cdot V = 0$. By Lemma 15 the distance $w = w(t,r)$ between x_o and an arbitrary point $x = \exp T \cdot \operatorname{Exp} V$ of y is simply given by

$$\cosh w(t,r) = \cosh t \cosh r, \quad t = \|T\|, \quad r = \|V\|, \tag{16}$$

the same expression as for real hyperbolic spaces.

According to (13) and (14) we only need to invert R at the origin for a K-invariant function u. As shown in Section 4.1 **a**, Lemma 1 applies and $Ru(y) = \int_y u(x)\, dm_y$. When u is radial the integral can be obtained in spherical coordinates on y with origin $\operatorname{Exp} T$, as

$$Ru(y) = \int_0^\infty u(w(t,r)) A_o(r)\, dr \tag{17}$$

where $A_o(r) = \omega_d (\sinh r)^{d-1} (\cosh r)^{q'}$ is the area of spheres of radius r in $\mathcal{Y}$. By (16) and (17) Ru may be viewed as a smooth even function $Ru(t)$ of $t \in \mathbf{R}$.

The end of the proof is now similar to the case of $H^n(\mathbf{R})$, as given in [11], p. 53 or [14], p. 97. Let $\tau = (\cosh t)^{-1}$, and let $t = t(\tau) \geq 0$ denote the inverse function. Introducing the functions

$$\varphi(\tau) = \tau^{-d-q'} u(t(\tau)), \quad \psi(\tau) = \tau^{-1-q'} Ru(t(\tau)),$$

which are C^∞ on $]0,1]$, (17) becomes

$$\psi(\tau) = \omega_d \int_0^\tau \varphi(\rho) \left(\tau^2 - \rho^2\right)^{(d/2)-1} d\rho. \tag{18}$$

Proof of (i). The Abel type integral equation (18) can be inverted as usual: it implies that, for any $a > 0$, $\sigma > 0$,

$$\Gamma\left(\frac{d}{2} + a\right) \int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau =$$

$$= \pi^{d/2} \Gamma(a) \int_0^\sigma \varphi(\rho) \left(\sigma^2 - \rho^2\right)^{(d/2)+a-1} d\rho$$

and, choosing $a > 0$ such that $N = (d/2) + a$ is a strictly positive integer, it follows easily that

$$2^{N-1} \pi^{d/2} \Gamma(a) \varphi(\sigma) = \sigma \left(\sigma^{-1} \partial_\sigma\right)^N \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau\right).$$

If $d = 2k+1$ is odd, $k \geq 0$, the smallest such a is $1/2$ so that $N = k+1$ and

$$2^k \pi^{k+1} \varphi(\sigma) = \sigma \left(\sigma^{-1} \partial_\sigma\right)^{k+1} \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{-1/2} \tau d\tau\right), \quad \sigma > 0;$$

the derivatives cannot be taken here under the integral. Besides d can only be odd for $\mathbf{F} = \mathbf{R}$ according to the assumption on $\mathfrak{s}$, and $q' = 0$ in that case. Going back to u and Ru we thus obtain for $\sigma = 1$

$$2^k \pi^{k+1} u(x_o) = \left(\sigma^{-1} \partial_\sigma\right)^{k+1} \int_0^\sigma Ru(t(\tau))(\sigma^2 - \tau^2)^{-1/2} d\tau \bigg|_{\sigma=1},$$

for any K-invariant $u \in \mathcal{D}(X)$. The claim follows by Section 6.2.

Proof of (ii). If $d = 2k$ is even, $k \geq 1$, the integral equation (18) can be directly solved as

$$(2\pi)^k \varphi(\tau) = \tau \left(\tau^{-1} \partial_\tau\right)^k \psi(\tau), \quad \tau > 0.$$

In particular, at the origin,

$$(2\pi)^k u(x_o) = \left(\tau^{-1}\partial_\tau\right)^k \left(\tau^{-1-q'} Ru(t(\tau))\right)_{\tau=1}$$
$$= \left(\partial_\tau^k + \cdots + (-1)^k (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t(\tau))\Big|_{\tau=1}.$$

To switch over to derivatives with respect to t we note that, if $g(\tau) = f(t)$ with $\tau = (\cosh t)^{-1} = 1 - \frac{t^2}{2} + \cdots$, identification of Taylor expansions at $\tau = 1$, resp. $t = 0$, leads to

$$\left(-\frac{1}{2}\right)^k \frac{g^{(k)}(1)}{k!} = \frac{f^{(2k)}(0)}{(2k)!} + \cdots + a_k f''(0),$$

where dots are a sum of even derivatives of f multiplied by some rational coefficients (like a_k). Therefore

$$(-2\pi)^k u(x_o) = \left(\frac{2^k k!}{(2k)!}\partial_t^{2k} + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t)|_{t=0},$$

for any K-invariant $u \in \mathcal{D}(X)$, whence the claim by Section 6.2. $\square$

6.4 The amusing formula generalized

a. To motivate the forthcoming generalizations of the amusing formula (12) and their applications to Radon inversion, we briefly recall the classical example of points and hyperplanes in the Euclidean space $X = \mathbf{R}^n$. Let (ω, p) be parameters for the hyperplane defined by the equation $\omega \cdot x = p$, where ω is a unit vector, p is a real number and $\cdot$ is the scalar product. Given $t \in \mathbf{R}$ and a point $x \in \mathbf{R}^n$, the parameters $(\omega, p) = (\omega, t + \omega \cdot x)$ define a hyperplane at distance $|t|$ from x, and

$$R_t^* v(x) = \int_{S^{n-1}} v(\omega, t + \omega \cdot x)\, d\omega$$

is the corresponding shifted dual Radon transform, where $v(\omega, p) = v(-\omega, -p)$ is an arbitrary smooth even function on $S^{n-1} \times \mathbf{R}$. Changing ω into $-\omega$ in the integral shows that $R_t^* v(x)$ is an even function of t.

Since $\sum \omega_i^2 = 1$ it is easily checked that

$$\left(\partial_t^2 - \Delta_x\right) v(\omega, t + \omega \cdot x) = 0,$$

where Δ_x is the Euclidean Laplace operator acting on x. Thus $R_t^* v(x)$, as a function of (x, t) in $\mathbf{R}^n \times \mathbf{R}$, is a solution of the wave equation, being an