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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

Autor: Rouvière, François

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Proof. Immediate, since

$$(R_t^* Ru)(g\cdot x_o) = \int_{K\times H} u(gkth\cdot x_o)\, dk\, dh = \int_H u_g(th\cdot x_o)\, dh\,. \qquad \square$$

Before proceeding we mention the following extension of Proposition 3 to shifted transforms. This result will not be used in the sequel.

PROPOSITION 12. *Let G and H be unimodular, K compact, $X = G/K$ and $Y = G/H$. For any $u \in C_c(X)$ and $t \in G$ we have*

$$R_t^* Ru = u * S_t$$

(convolution on X). Here S_t is the K-invariant distribution on X defined by $S = R_t^* R\delta$, and δ is the Dirac distribution at the origin $x_o = K$ of X, i.e.

$$\langle S_t, u\rangle = R^* R_t u(x_o) = \int_{K\times H} u(kht^{-1}\cdot x_o)\, dk\, dh\,.$$

Proof. The proof of Proposition 3 can be repeated here, with $R^* R_t$ as the dual of $R_t^* R$. The claim can also be checked directly, writing, for $\varphi \in \mathcal{D}(X)$,

$$\langle R_t^* Ru, \varphi\rangle = \int_{G\times H} u(gth\cdot x_o)\varphi(g\cdot x_o)\, dg\, dh\,,$$

and changing variables into $h' = h^{-1}$, $g' = gth$; the result follows easily, G and H being unimodular groups. Details are left to the reader. $\square$

6.2 RADON INVERSION BY SHIFTS

The elementary Lemma 11 can be used in the following way. Assume the transform R can be inverted at the origin for K-invariant functions on X, say

$$u(x_o) = \langle T_{(y)}, Ru(y)\rangle\,, \tag{13}$$

where T is some linear form on a space of functions on Y. Then, replacing u by the K-invariant function u_g in the lemma, we obtain

$$u(g\cdot x_o) = u_g(x_o) = \langle T, Ru_g\rangle\,.$$

The roles of g and t can now be interchanged by Lemma 11, whence

$$u(x) = \langle T_{(t)}, R_t^* Ru(x)\rangle\,, \tag{14}$$

for arbitrary $u \in \mathcal{D}(X)$ and $x \in X$. The notation $T_{(t)}$ means that T now acts on the shift variable t, or $t\cdot y_o$ to be precise. Since $R_{kth}^* Ru(x) = R_t^* Ru(x)$ for $k \in K$ and $h \in H$, this variable may actually be taken in $K\backslash G/H$.

The general inversion formula (14) *for* R *thus follows from the special case* (13) *of* K*-invariant functions at the origin*, thanks to the shifted dual transform.

If X is an isotropic space, the above trick (replace u by u_g) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

6.3 Examples

a. Horocycle transform. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, **d**), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in X, resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o)\, dn$$

(integrating over M is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o)\, dk\,.$$

For K-invariant u the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o)\, dn = a^{-\rho} \mathcal{A}u(a)\,;$$

the *Abel transform* $\mathcal{A}$ is defined by this equality.

For K-invariant $u \in \mathcal{D}(X)$ we have $\mathcal{A}u \in \mathcal{D}(A)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. It is known from spherical harmonic analysis on X that the classical Fourier transform

$$\widehat{\mathcal{A}u}(\lambda) = \int_A a^{-i\lambda} \mathcal{A}u(a)\, da\,, \quad \lambda \in \mathfrak{a}^*\,,$$

coincides with the spherical transform of u, with the inversion formula ([9] p. 454)

$$u(x_o) = C \int_{\mathfrak{a}^*} \widehat{\mathcal{A}u}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda\,, \tag{15}$$

where C is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since