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Autor: Rouvière, François
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Proof. Immediate, since

$$(R_t^* Ru)(g \cdot x_o) = \int_{K \times H} u(gkth \cdot x_o) dk dh = \int_H u_g(th \cdot x_o) dh. \quad \square$$

Before proceeding we mention the following extension of Proposition 3 to shifted transforms. This result will not be used in the sequel.

PROPOSITION 12. *Let G and H be unimodular, K compact, $X = G/K$ and $Y = G/H$. For any $u \in C_c(X)$ and $t \in G$ we have*

$$R_t^* Ru = u * S_t$$

(convolution on X). Here S_t is the K -invariant distribution on X defined by $S = R_t^* R \delta$, and δ is the Dirac distribution at the origin $x_o = K$ of X , i.e.

$$\langle S_t, u \rangle = R^* R_t u(x_o) = \int_{K \times H} u(kht^{-1} \cdot x_o) dk dh.$$

Proof. The proof of Proposition 3 can be repeated here, with $R^* R_t$ as the dual of $R_t^* R$. The claim can also be checked directly, writing, for $\varphi \in \mathcal{D}(X)$,

$$\langle R_t^* Ru, \varphi \rangle = \int_{G \times H} u(gth \cdot x_o) \varphi(g \cdot x_o) dg dh,$$

and changing variables into $h' = h^{-1}$, $g' = gth$; the result follows easily, G and H being unimodular groups. Details are left to the reader. \square

6.2 RADON INVERSION BY SHIFTS

The elementary Lemma 11 can be used in the following way. Assume the transform R can be inverted at the origin for K -invariant functions on X , say

$$(13) \quad u(x_o) = \langle T_{(y)}, Ru(y) \rangle,$$

where T is some linear form on a space of functions on Y . Then, replacing u by the K -invariant function u_g in the lemma, we obtain

$$u(g \cdot x_o) = u_g(x_o) = \langle T, Ru_g \rangle.$$

The roles of g and t can now be interchanged by Lemma 11, whence

$$(14) \quad u(x) = \langle T_{(t)}, R_t^* Ru(x) \rangle,$$

for arbitrary $u \in \mathcal{D}(X)$ and $x \in X$. The notation $T_{(t)}$ means that T now acts on the shift variable t , or $t \cdot y_o$ to be precise. Since $R_{kth}^* Ru(x) = R_t^* Ru(x)$ for $k \in K$ and $h \in H$, this variable may actually be taken in $K \backslash G/H$.

The general inversion formula (14) for R thus follows from the special case (13) of K -invariant functions at the origin, thanks to the shifted dual transform.

If X is an isotropic space, the above trick (replace u by u_g) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

6.3 EXAMPLES

a. HOROCYCLE TRANSFORM. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, **d**), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in X , resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o) dn$$

(integrating over M is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o) dk.$$

For K -invariant u the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o) dn = a^{-\rho} Au(a);$$

the Abel transform \mathcal{A} is defined by this equality.

For K -invariant $u \in \mathcal{D}(X)$ we have $Au \in \mathcal{D}(A)$. Let \mathfrak{a}^* be the dual space of \mathfrak{a} . It is known from spherical harmonic analysis on X that the classical Fourier transform

$$\widehat{Au}(\lambda) = \int_A a^{-i\lambda} Au(a) da, \quad \lambda \in \mathfrak{a}^*,$$

coincides with the spherical transform of u , with the inversion formula ([9] p.454)

$$(15) \quad u(x_o) = C \int_{\mathfrak{a}^*} \widehat{Au}(\lambda) |c(\lambda)|^{-2} d\lambda,$$

where C is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since